B5.3 Viscous Flow: Sheet 2

Q1 **Unidirectional flow.**

(a) Show that for unidirectional flow with velocity $\mathbf{u} = u(x, y, z, t)\mathbf{i}$ the incompressible Navier-Stokes equations with no body forces become

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x}\right) = -\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right), \quad 0 = -\frac{\partial p}{\partial y}, \quad 0 = -\frac{\partial p}{\partial z}, \quad \frac{\partial u}{\partial x} = 0$$

in the absence of external body forces.

(b) Use a separation-of-variables type argument to deduce that the pressure gradient $\partial p/\partial x = G(t)$, where $G(t)$ is an arbitrary function of time t. Hence show that $u(y, z, t)$ satisfies the two-dimensional diffusion equation

$$\frac{\partial u}{\partial t} = \nu\left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) - \frac{G(t)}{\rho}, \tag{1}$$

where $\nu = \mu/\rho$ is the coefficient of kinematic viscosity.

(c) Determine the vorticity $\boldsymbol{\omega} = \boldsymbol{\nabla} \wedge \mathbf{u}$ and use (1) to show that it satisfies the two-dimensional diffusion equation

$$\frac{\partial \boldsymbol{\omega}}{\partial t} = \nu\left(\frac{\partial^2 \boldsymbol{\omega}}{\partial y^2} + \frac{\partial^2 \boldsymbol{\omega}}{\partial z^2}\right).$$

Q2 **Steady Couette/Poiseuille flow in a channel.** Incompressible Newtonian fluid flows *steadily* between two parallel rigid plates at $y = 0$ and $y = h$. The bottom plate moves in the x-direction with velocity U and the top plate moves in the x-direction with velocity V. There are no external body forces. The flow is unidirectional with velocity $\mathbf{u} = u(y)\mathbf{i}$ and there is a constant applied pressure gradient $\partial p/\partial x = G < 0$.

(a) Use (1) to show that $u(y)$ satisfies the ordinary differential equation

$$\frac{\mathrm{d}^2 u}{\mathrm{d}y^2} = \frac{G}{\mu}.$$

Use the no-slip condition on each of the plates to show that $u(0) = U$ and $u(h) = V$. Hence find $u(y)$.

(b) Calculate the net volume flux, *i.e.* the volume of fluid flowing through a plane $x =$ constant per unit length in z and per unit time,

$$Q = \int_0^h u(y)\,\mathrm{d}y.$$

(c) Calculate the shear stress

$$\sigma_{12} = \mu\frac{\mathrm{d}u}{\mathrm{d}y}.$$

Hence find the component of force acting on each of the plates in the x-direction per unit area.

(d) Suppose that $U = V$. For what boundary speeds U is the net volume flux zero? With this value of U, sketch the velocity profile. For what boundary speeds U do all fluid particles move in the *negative* x-direction?

Q3 **Poiseuille flow in a pipe.** Incompressible Newtonian fluid flows steadily down a straight cylindrical pipe of uniform cross-section D. The generators of the pipe are parallel to the x-axis and the flow is unidirectional with velocity $\mathbf{u} = u(y, z)\mathbf{i}$. There is are no external body forces and there is a constant applied pressure gradient $\partial p/\partial x = G$.

(a) Use (1) to show that $u(y, z)$ satisfies Poisson's equation

$$\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{G}{\mu} \text{ in } D,$$

with $u = 0$ on the boundary ∂D.

(b) Evaluate the stress tensor σ_{ij} in terms of $u(y, z)$ and $p(x)$ and deduce that the drag (*i.e.* the component of force acting on the wall in the x-direction per unit length in x) is given by

$$-\int_{\partial D} \mu\mathbf{n}\cdot\boldsymbol{\nabla}u\,\mathrm{d}s,$$

where $\mathbf{n} = (0, n_2, n_3)$ is the unit *outward* normal and $\mathrm{d}s$ is an element of arc length. Hence, or otherwise, show that the drag is equal to $-G$ times the area of the cross-section.

(c) (i) If the flow is axisymmetric, with $u = u(r)$ in cylindrical polar coordinates (r, θ), show that

$$\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}u}{\mathrm{d}r}\right) = \frac{G}{\mu} \text{ in } D.$$

(ii) If D is a circle of radius a, write down the drag and show that the velocity $u(r)$ and volume flux Q down the pipe are given by

$$u(r) = -\frac{G}{4\mu}\left(a^2 - r^2\right), \quad Q = \iint_D u(y, z)\,\mathrm{d}y\mathrm{d}z = -\frac{\pi a^4 G}{8\mu}.$$

Q4 **Stokes layer.** Incompressible Newtonian fluid occupies the region $y > 0$ above a rigid plate at $y = 0$ which oscillates to and fro in the x-direction with velocity $U \cos \Omega t$. There are no external body forces and there is no applied pressure gradient. The flow is unidirectional with velocity $\mathbf{u} = u(y, t)\mathbf{i}$.

(a) Use (1) to show that $u(y, t)$ satisfies the one-dimensional diffusion equation

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2}, \tag{2}$$

with $u(0, t) = U \cos \Omega t$ and $u(\infty, t) = 0$ for $-\infty < t < \infty$.

(b) By seeking a solution of the form $u = \Re(U f(y) e^{i\Omega t})$, where $\Re$ means 'real part', show that

$$u = U\mathrm{e}^{-ky}\cos(ky - \Omega t),$$

where $k = (\Omega/2\nu)^{1/2}$.

(c) Determine the vorticity $\boldsymbol{\omega} = \boldsymbol{\nabla} \wedge \mathbf{u}$ and show that its magnitude is exponentially small *except* in a layer of thickness of order $(\nu/\Omega)^{1/2}$, independent of time t. Sketch the velocity profile at time $t = 0$, indicating the (Stokes) layer in which the vorticity is significant.

Q5 **Rayleigh layer.** Incompressible Newtonian fluid lies at rest in the region $y > 0$ above a rigid plate at $y = 0$. At time $t = 0$ the plate is jerked into motion in the x-direction with constant velocity U. There are no external body forces and there is no applied pressure gradient. The flow is unidirectional with velocity $\mathbf{u} = u(y, t)\mathbf{i}$, so that $u(y, t)$ satisfies the one-dimensional diffusion equation (2).

(a) Explain why $u(0, t) = U$, $u(\infty, t) = 0$ for $t > 0$ and $u(y, 0) = 0$ for $y > 0$.

(b) Show that there is a similarity solution of the form $u(y, t) = U f(\eta)$, $\eta = y/(4\nu t)^{1/2}$, where

$$f'' + 2\eta f' = 0,$$

with $f(0) = 1$, $f(\infty) = 0$. Hence show that $u(y, t) = U \operatorname{erfc}\left(y/(4\nu t)^{1/2}\right)$, where

$$\operatorname{erfc}(\eta) = \frac{2}{\sqrt{\pi}}\int_\eta^\infty \exp(-s^2)\,\mathrm{d}s$$

and you should assume that $\operatorname{erfc}(0) = 1$.

(c) Determine the vorticity $\boldsymbol{\omega} = \boldsymbol{\nabla} \wedge \mathbf{u}$ and show that its magnitude is exponentially small *except* in a layer of thickness of order $(\nu t)^{1/2}$. Sketch the velocity profile, indicating the (Rayleigh) layer in which the vorticity is significant.

Q6 **Nondimensionalization and the Reynolds number.**

(a) Consider flow past an obstacle of typical size L, with speed U far away. Explain why ρU^2 is a possible pressure scale and make dimensionless both the incompressible Navier-Stokes equations with no body forces and the vorticity-transport equation in sheet 1, Q5(b). Show that there is just one dimensionless paramater: the *Reynolds number* $Re = UL/\nu$.

(b) Explain the physical significance of the Reynolds number in terms of the ratio of (i) inertia to viscous forces and (ii) the timescales for diffusion and convection of vorticity.

(c) By considering the right-hand side of the momentum equation, show that $\mu U/L$ is another possible scale for the pressure and make the equations dimensionless with this scale. Which non-dimensional form might you use if Re is (i) large, (ii) small?

(d) Under what conditions are two flows *dynamically similar*?

(e) [1]Using the order of magnitude estimates of the kinematic viscosity at normal temperatures listed below, give an order of magnitude estimate of the Reynolds number for the following situations:

(i) a pebble thrown into a pond;
(ii) a bubble rising in a glass of champagne;
(iii) pouring golden syrup;
(iv) the formation of a tear drop (crying);
(v) a layer of freshly applied paint;
(vi) oil in a car engine;
(vii) water permeating sand on a beach;
(viii) oil in an oil well;
(ix) rock convecting in the earth's mantle.

Liquid	Kinematic viscosity ν
water, tears	10^{-6} m^{2}s^{-1}
air	10^{-5} m^{2}s^{-1}
engine oil	10^{-4} m^{2}s^{-1}
paint	10^{-3} m^{2}s^{-1}
crude oil	10^{-5} m^{2}s^{-1}
syrup	10^{-1} m^{2}s^{-1}
mantle rock	10^{18} m^{2}s^{-1}

[1]This is not a mathematical question, but it has mathematical implications! This part of the question will not be marked, but please try all of (i)–(ix) ready for a discussion with your class.