B5.3 Viscous Flow: Sheet 3

Q1 Thermal boundary layer on a semi-infinite flat plate. Consider the two-dimensional steady heat convection-conduction problem in which *inviscid* fluid with constant velocity $U\mathbf{i}$ and temperature T_∞ flows past a 'hot' semi-infinite plate at $y = 0$, $x > 0$, which is held at constant temperature T_p. Assume that the density ρ, specific heat c_v and thermal conductivity k are constant.

(a) Starting from the conservation of energy equation in sheet 1, Q6(b) show that the temperature $T(x, y)$ satisfies

$$U\frac{\partial T}{\partial x} = \kappa\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right),$$

where $\kappa = k/\rho c_v$ is the constant thermal diffusivity. By using the dimensionless variables

$$x^* = \frac{x}{L},\ y^* = \frac{y}{L},\ T^* = \frac{T - T_\infty}{T_p - T_\infty},$$

where L is an arbitrary length scale, rewrite the problem in dimensionless form (dropping the stars * on the dimensionless variables):

$$\frac{\partial T}{\partial x} = \frac{1}{Pe}\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right),$$

with $T = 1$ on $y = 0$, $x > 0$ and $T \to 0$ as $x^2 + y^2 \to \infty$. Explain the physical significance of the *Péclet number* $Pe = LU/\kappa$ in terms of the timescales for conduction and convection of heat.

(b) *Given* that it is possible to find a similarity solution in the form $T(x, y) = f(\eta)$, where $x + iy = (\xi + i\eta)^2/Pe$ and $f(\eta)$ satisfies

$$\text{for } \eta > 0,\ f'' + 2\eta f' = 0;\quad f(0) = 1,\ f(\infty) = 0,$$

show that $T(x, y) = \operatorname{erfc}(\eta)$. Deduce that the isotherms are parabolic and indicate on a diagram the regions of the (x, y)-plane where $T = O(1)$ as $Pe \to \infty$.

(c) Deduce from the governing equations that for $Pe \gg 1$ there is a boundary layer on the plate in which $Y = Pe^{1/2}y = \mathrm{O}(1)$ and $T \sim T_0(x, Y)$, where

$$\frac{\partial T_0}{\partial x} = \frac{\partial^2 T_0}{\partial Y^2}, \tag{1}$$

with $T_0(x, 0) = 1$, $T_0(x, \infty) = 0$ for $x > 0$. Hence show that $T_0 = \operatorname{erfc}\left(Y/(4x)^{1/2}\right)$.

(d) Finally, show that the exact and asymptotic solution are in agreement in the boundary layer, *i.e.* show that $T(x, Pe^{-1/2}Y) \sim T_0(x, Y)$ as $Pe \to \infty$, with $Y = \mathrm{O}(1)$.

Q2 High-Reynolds number flow past a semi-infinite flat plate. Consider the two-dimensional steady *viscous* flow of a uniform stream with velocity $U\mathbf{i}$ past a semi-infinite plate at $y = 0$, $x > 0$.

(a) Starting from the vorticity-streamfunction formulation in sheet 1, Q5(c)(ii) show that the dimensionless problem for the streamfunction $\psi(x, y)$ is given by

$$\frac{\partial(\psi, \nabla^2\psi)}{\partial(y, x)} = \frac{1}{Re}\nabla^4\psi, \tag{2}$$

with (upon taking ψ to be equal to zero on the plate)

$$\psi = \frac{\partial\psi}{\partial y} = 0 \text{ on } y = 0,\ x > 0;\quad \frac{\partial\psi}{\partial y} \to 1 \text{ as } x^2 + y^2 \to \infty, \tag{3}$$

where the dimensionless variables x, y, ψ and the Reynolds number Re should be defined.

(b) When $Re = \infty$, show that $\psi = y$ satisfies (**??**) and (**??**) *except* for the no-slip condition. When Re is large but finite, show that there is a boundary layer on the plate in which $Y = Re^{1/2}y = \mathrm{O}(1)$ and $\psi \sim Re^{-1/2}\Psi$, where $\Psi(x, Y)$ satisfies the boundary layer equation

$$\frac{\partial\Psi}{\partial Y}\frac{\partial^3\Psi}{\partial x\partial Y^2} - \frac{\partial\Psi}{\partial x}\frac{\partial^3\Psi}{\partial Y^3} = \frac{\partial^4\Psi}{\partial Y^4},$$

together with the boundary and matching conditions

$$\Psi = \frac{\partial\Psi}{\partial Y} = 0 \text{ on } Y = 0,\ x > 0;\quad \frac{\partial\Psi}{\partial Y} \to 1 \text{ as } Y \to \infty.$$

(c) Deduce that

$$\frac{\partial \Psi}{\partial Y}\frac{\partial^2 \Psi}{\partial x \partial Y} - \frac{\partial \Psi}{\partial x}\frac{\partial^2 \Psi}{\partial Y^2} = \frac{\partial^3 \Psi}{\partial Y^3}, \tag{4}$$

and hence show that there is a similarity solution of the form $\Psi(x, Y) = x^\alpha f(\eta)$, $Y = x^\beta \eta$ provided $\alpha = \beta = 1/2$ and $f(\eta)$ satisfies Blasius' equation

$$f''' + \frac{1}{2} f f'' = 0,$$

with $f(0) = f'(0) = 0$ and $f'(\infty) = 1$.

Q3 Viscous boundary layer with a non-uniform slip velocity. An incompressible Newtonian fluid flows past a solid boundary which lies on the positive x-axis. The flow is two-dimensional and governed by the dimensionless steady incompressible Navier-Stokes equations

$$(\mathbf{u} \cdot \boldsymbol{\nabla})\mathbf{u} = -\boldsymbol{\nabla} p + \frac{1}{Re}\boldsymbol{\nabla}^2 \mathbf{u}, \ \boldsymbol{\nabla} \cdot \mathbf{u} = 0, \tag{5}$$

where $\mathbf{u} = u(x, y)\mathbf{i} + v(x, y)\mathbf{j}$ is the velocity, $p(x, y)$ is the pressure and Re is the Reynolds number. Suppose that when $Re = \infty$, the external *inviscid irrotational* flow generates a non-uniform slip velocity $U_s(x)$ on the plate.

(a) Show that, when Re is large but finite, the flow near the plate only differs appreciably from $U_s(x)$ in a boundary layer in which $Y = Re^{1/2} y = \mathrm{O}(1)$, $v \sim Re^{-1/2} V(x, Y)$ and Prandtl's boundary layer equations

$$u\frac{\partial u}{\partial x} + V\frac{\partial u}{\partial Y} = -\frac{\partial p}{\partial x} + \frac{\partial^2 u}{\partial Y^2}, \ 0 = -\frac{\partial p}{\partial Y}, \ \frac{\partial u}{\partial x} + \frac{\partial V}{\partial Y} = 0$$

pertain. Explain briefly why the boundary and far-field matching conditions are given by

$$u = V = 0 \ \text{ on } \ Y = 0, \ x > 0; \quad u \to U_s(x) \ \text{ as } \ Y \to \infty,$$

and deduce that the pressure gradient $\partial p/\partial x = -U_s(x)U_s'(x)$.

(b) Show that there is a streamfunction $\Psi(x, Y)$ satisfying

$$\frac{\partial \Psi}{\partial Y}\frac{\partial^2 \Psi}{\partial x \partial Y} - \frac{\partial \Psi}{\partial x}\frac{\partial^2 \Psi}{\partial Y^2} = \frac{\partial^3 \Psi}{\partial Y^3} + U_s(x)U_s'(x), \tag{6}$$

and write down the boundary conditions for Ψ.

(c) Suppose there is a similarity solution of the form

$$\Psi(x, Y) = U_s(x)g(x)f(\eta), \ Y = g(x)\eta.$$

(i) Show that the boundary layer equation (**??**) becomes

$$f'''(\eta) + \alpha(x)f(\eta)f''(\eta) + \beta(x)(1 - f'(\eta)^2) = 0,$$

where $\alpha(x) = g(x)\big(g(x)U_s(x)\big)'$ and $\beta(x) = g(x)^2 U_s'(x)$. Explain why both α and β must be constant.

(ii) Find α, β and $g(x)$ when $U_s(x) = x^m$ and $g(1) = 1$, and hence write down the Falkner-Skan equation for $f(\eta)$. What are the boundary conditions for $f(\eta)$? How might a slip velocity $U_s(x) \propto x^m$ arise in practice?

Q4 High-Reynolds number Jeffery-Hamel flow. In the absence of body forces and in plane polar coordinates (r, θ) the steady Navier-Stokes equations for an incompressible Newtonian fluid with uniform density ρ and kinematic viscosity ν are given by

$$\begin{aligned}
u_r\frac{\partial u_r}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta^2}{r} &= -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_r}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u_r}{\partial \theta^2} - \frac{u_r}{r^2} - \frac{2}{r^2}\frac{\partial u_\theta}{\partial \theta}\right), \\
u_r\frac{\partial u_\theta}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r u_\theta}{r} &= -\frac{1}{\rho r}\frac{\partial p}{\partial \theta} + \nu\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_\theta}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u_\theta}{\partial \theta^2} - \frac{u_\theta}{r^2} + \frac{2}{r^2}\frac{\partial u_r}{\partial \theta}\right), \\
\frac{1}{r}\frac{\partial}{\partial r}(r u_r) + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} &= 0,
\end{aligned}$$

where $\mathbf{u} = u_r(r, \theta)\mathbf{e}_r + u_\theta(r, \theta)\mathbf{e}_\theta$ is the velocity, p is the pressure and $\mathbf{e}_r$, $\mathbf{e}_\theta$ are unit vectors in the r- and θ-directions. Radial flow is generated in a wedge $-\alpha < \theta < \alpha$ by a source ($Q > 0$) or sink ($Q < 0$) of strength Q at the origin.

(a) Show that $u_r = |Q| g(\theta)/r$, where the dimensionless function $g(\theta)$ satisfies

$$g''' + 4g' + 2Re\, gg' = 0,$$

with $g(-\alpha) = g(\alpha) = 0$ and

$$\int_{-\alpha}^{\alpha} g(\theta)\,\mathrm{d}\theta = \mathrm{sgn}(Q),$$

where the Reynolds number $Re = |Q|/\nu$.

(b) Suppose the Reynolds number is *large* (*i.e.* $Re \gg 1$) and that the effects of viscosity are confined to boundary layers on the walls.

(i) In the outer region away from the walls, show that $g \sim \mathrm{sgn}(Q)/2\alpha$ as $Re \to \infty$.

(ii) In the boundary layer on the wall at $\theta = -\alpha$ in which $\phi = Re^{1/2}(\alpha + \theta) = \mathrm{O}(1)$, show that $g \sim G$, where $G(\phi)$ satisfies

$$\frac{\mathrm{d}^2 G}{\mathrm{d}\phi^2} + G^2 = \frac{1}{4\alpha^2},$$

with $G(0) = 0$ and $G(\infty) = \mathrm{sgn}(Q)/2\alpha$.

(iii) Deduce that such a solution is only possible for in-flow (*i.e.* $Q < 0$).