B5.3 Viscous Flow: Sheet 4

Q1 Slow flow past a circular cylinder. Consider the two-dimensional steady viscous flow of a uniform stream with velocity $U\mathbf{i}$ past a rigid circular cylinder of radius a whose centre is at the origin of the plane polar coordinate system (r, θ).

(a) By scaling $(r,\ \psi,\ \omega)$ with $(a,\ Ua,\ U/a)$ in the vorticity–streamfunction formulation in sheet 1, Q5(c)(iii), show that the dimensionless problem is given by

$$Re\,\frac{1}{r}\frac{\partial(\psi,\omega)}{\partial(\theta,r)} = \boldsymbol{\nabla}^2\omega,\ \ -\omega = \boldsymbol{\nabla}^2\psi \equiv \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2}.$$

with (upon taking ψ to be equal to zero on the cylinder)

$$\psi = \frac{\partial\psi}{\partial r} = 0 \text{ on } r = 1; \quad \psi \sim r\sin\theta \text{ as } r \to \infty,$$

where the Reynolds number $Re = Ua/\nu$.

(b) When the Reynolds number is *small* (*i.e.* $Re \ll 1$), show that the slow flow approximation leads to

$$\boldsymbol{\nabla}^4\psi = 0,$$

and by separating the variables as $\psi = f(r)\sin\theta$ show that

$$f = \frac{A}{r} + Br + Cr\log r + Dr^3.$$

(c) Write down the boundary conditions which f must satisfy at $r = 1$ and show that if f satisfies these conditions then ψ cannot approach the free stream at infinity. Explain, without detailed calculations, how this paradox is resolved. Given that the resolution of the paradox leads to $C = 1/\ln(1/Re)$, $D = 0$, use the remaining boundary conditions to determine A and B.

Q2 Slow flow past a sphere. Incompressible Newtonian fluid flows with constant velocity $U\mathbf{k}$ past a sphere of radius a whose centre is at the origin of the spherical polar coordinate system (r, θ, ϕ).

(a) Starting from the steady incompressible Navier-Stokes equations with no body forces, explain how the *dimensionless* slow flow approximation

$$(\text{curl})^3\mathbf{u} = \mathbf{0}, \qquad \boldsymbol{\nabla}\cdot\mathbf{u} = 0 \tag{1}$$

can be determined when the Reynolds number is small.

(b) An axisymmetric solution of (**??**) can be written as

$$\mathbf{u} = \text{curl}\left(\frac{\psi}{r\sin\theta}\mathbf{e}_\phi\right) = \frac{1}{r^2\sin\theta}\frac{\partial\psi}{\partial\theta}\mathbf{e}_r - \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial r}\mathbf{e}_\theta,$$

where $\mathbf{e}_r$, $\mathbf{e}_\theta$ and $\mathbf{e}_\phi$ are unit vectors in the r-, θ- and ϕ-directions. Given that

$$\text{curl}^2\left(\frac{\psi}{r\sin\theta}\mathbf{e}_\phi\right) = -\frac{D^2\psi}{r\sin\theta}\mathbf{e}_\phi \quad \text{where} \quad D^2 = \frac{\partial^2}{\partial r^2} - \frac{\cot\theta}{r^2}\frac{\partial}{\partial\theta} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2},$$

show that ψ satisfies

$$D^4\psi = 0 \text{ for } r > 1; \quad \psi = \frac{\partial\psi}{\partial r} = 0 \text{ on } r = 1; \quad \psi \sim \frac{1}{2}r^2\sin^2\theta \text{ as } r \to \infty.$$

(c) By separating the variables as $\psi = f(r)\sin^2\theta$, show that

$$\psi = \left(\frac{1}{2}r^2 - \frac{3}{4}r + \frac{1}{4}r^{-1}\right)\sin^2\theta.$$

Q3 Lubrication theory for a slider-bearing. Incompressible Newtonian fluid occupies the thin gap, of thickness $O(\delta L)$, between a flat plate $z = 0$ moving with constant velocity U in the x-direction and a stationary rigid surface described by $z = h(x)$, $0 < x < L$. There is no gravitational field and the ends of the slider-bearing are held at the ambient pressure p_{atm}. Assume that the flow is two-dimensional with velocity $\mathbf{u} = u(x,z)\mathbf{i} + w(x,z)\mathbf{k}$ and pressure $p(x,z)$, and governed by the incompressible Navier-Stokes equations with no body forces.

(a) By using the dimensionless variables

$$x^* = \frac{x}{L},\; z^* = \frac{z}{\delta L},\; h^* = \frac{h}{\delta L},\; u^* = \frac{u}{U},\; w^* = \frac{w}{\delta U},\; p^* = \frac{p - p_{atm}}{\mu U/\delta^2 L},$$

show that the Navier-Stokes equations become (dropping the stars * on the dimensionless variables)

$$\begin{aligned} \delta^2 Re\left(u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z}\right) &= -\frac{\partial p}{\partial x} + \delta^2\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial z^2}, \\ \delta^4 Re\left(u\frac{\partial w}{\partial x} + w\frac{\partial w}{\partial z}\right) &= -\frac{\partial p}{\partial z} + \delta^4\frac{\partial^2 w}{\partial x^2} + \delta^2\frac{\partial^2 w}{\partial z^2}, \\ \frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} &= 0, \end{aligned}$$

with $(u,w) = (1,0)$ on $z = 0$ and $(u,w) = (0,0)$ on $z = h(x)$ for $0 < x < 1$, where $Re = \rho UL/\mu$.

(b) Deduce that if $\delta \ll 1$ and $\delta^2 Re \ll 1$, then the lubrication equations

$$\frac{\partial^2 u}{\partial z^2} = \frac{\partial p}{\partial x}, \quad \frac{\partial p}{\partial z} = 0, \quad \frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0$$

pertain at leading order. Hence deduce Reynolds equation for $p(x)$ in the form

$$\frac{d}{dx}\left(h^3\frac{dp}{dx}\right) = 6\frac{dh}{dx}.$$

(c) In dimensionless variables a slider bearing has gap thickness $h(x) = (1 - \lambda x)$ for $0 < x < 1$ and $0 < \lambda < 1$. Calculate the pressure $p(x)$ within the bearing and show that the total load supported by the bearing per unit length in y is given by

$$\int_0^1 p(x)\,\mathrm{d}x = \frac{6}{\lambda^2}\left(\log\frac{1}{1-\lambda} - \frac{2\lambda}{2-\lambda}\right).$$

Q4 Injection problem in a Hele-Shaw cell. In a Hele-Shaw cell, a viscous fluid is injected with velocity U between two rigid parallel plates which are of lateral extent L and a fixed distance δL apart. There is no gravitational field.

(a) Starting from the incompressible Navier-Stokes equations with no body forces and the z-axis normal to the plates, show that, provided $\delta \ll 1$ and $\delta^2\rho UL/\mu \ll 1$, the flow satisfies at leading order the *dimensional* lubrication equations

$$0 = -\frac{\partial p}{\partial x} + \mu\frac{\partial^2 u}{\partial z^2}, \quad 0 = -\frac{\partial p}{\partial y} + \mu\frac{\partial^2 v}{\partial z^2}, \quad 0 = -\frac{\partial p}{\partial z}, \quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0,$$

with $u = v = w = 0$ on $z = 0$ and $z = h$.

(b) Hence show that, if $\bar{u}$, $\bar{v}$ are the *mean* velocities in the x- and y-directions respectively, then

$$(\bar{u},\bar{v}) = -\frac{h^2}{12\mu}\left(\frac{\partial p}{\partial x}, \frac{\partial p}{\partial y}\right) \quad \text{and} \quad \frac{\partial}{\partial x}\left(h\bar{u}\right) + \frac{\partial}{\partial y}\left(h\bar{v}\right) = 0.$$

Deduce that $p(x,y)$ satisfies Laplace's equation.

(c) A circular blob of fluid of radius R_0 centred at the origin is at rest within the cell. At $t = 0$, a source of constant strength Q is introduced at the origin so that in the subsequent flow the fluid is contained within a circle of radius $R(t)$. What are the boundary conditions on p and $\bar{\mathbf{u}} = (\bar{u},\bar{v})$ on $r = R(t)$?

(d) Show that a possible solution gives

$$R(t) = \sqrt{R_0^2 + \frac{Qt}{\pi h}}$$

and determine the corresponding pressure. Do you expect this solution to be valid for both $Q > 0$ and $Q < 0$?

Q5 Gravity-driven flow of a thin film. A thin two-dimensional sheet of viscous fluid lies on a plate which is at an angle α to the horizontal. Initially the sheet is of width L and maximum height δL where $\delta \ll 1$, and the fluid flows over the plate under gravity. Choose Cartesian coordinates (x, z) tangential and normal to the plate respectively, with x measured down the plate along the line of greatest slope. Denote by $\mathbf{u} = u(x, z, t)\mathbf{i} + w(x, z, t)\mathbf{k}$ the liquid velocity, by $p(x, z, t)$ the pressure and by $z = h(x, t)$ the free surface at which the kinematic and zero stress conditions pertain. The flow is governed by the incompressible Navier-Stokes equations with a body force due solely to the gravitational acceleration $\mathbf{F} = g(\mathbf{i}\sin\alpha - \mathbf{k}\cos\alpha)$.

(a) By using the dimensionless variables

$$x^* = \frac{x}{L},\ z^* = \frac{z}{\delta L},\ h^* = \frac{h}{\delta L},\ u^* = \frac{u}{U},\ w^* = \frac{w}{\delta U},\ t^* = \frac{t}{L/U},\ p^* = \frac{p}{\mu U/\delta^2 L},$$

where U is a representative velocity scale, show that the Navier-Stokes equations become (dropping the stars * on the dimensionless variables)

$$\begin{aligned} \delta^2 Re\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z}\right) &= -\frac{\partial p}{\partial x} + \delta^2\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial z^2} + \frac{\delta^2 \rho g L^2}{\mu U}\sin\alpha, \\ \delta^4 Re\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + w\frac{\partial w}{\partial z}\right) &= -\frac{\partial p}{\partial z} + \delta^4\frac{\partial^2 w}{\partial x^2} + \delta^2\frac{\partial^2 w}{\partial z^2} - \frac{\delta^3 \rho g L^2}{\mu U}\cos\alpha, \\ \frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} &= 0, \end{aligned}$$

where the Reynolds number $Re = \rho U L/\mu$.

(b) Show that the stress tensor is given by

$$\sigma_{11} \sim \sigma_{33} \sim -\frac{\mu U}{\delta^2 L}p, \quad \sigma_{13} = \sigma_{31} \sim \frac{\mu U}{\delta L}\frac{\partial u}{\partial z} \quad \text{as} \quad \delta \to 0$$

(c) If the plate angle $\alpha = \mathrm{O}(1)$ as $\delta \to 0$, show that, in the thin-film regime in which $\delta \ll 1$ and $\delta^2 Re \ll 1$, an appropriate velocity scale is $U = \delta^2 \rho g L^2 \sin\alpha/\mu$ and at leading order the lubrication equations

$$0 = -\frac{\partial p}{\partial x} + \frac{\partial^2 u}{\partial z^2} + 1, \quad 0 = -\frac{\partial p}{\partial z}, \quad \frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0$$

pertain, with

$$u = w = 0 \quad \text{on} \quad z = 0,$$

and

$$p = \frac{\partial u}{\partial z} = 0, \quad w = \frac{\partial h}{\partial t} + u\frac{\partial h}{\partial x} \quad \text{on} \quad z = h(x, t).$$

Hence show $h(x, t)$ satisfies the equation

$$\frac{\partial h}{\partial t} + h^2\frac{\partial h}{\partial x} = 0.$$

Verify that $h = f\big(x - h^2 t\big)$ solves this equation, so that particular values of h propagate down the plate with speed proportional to h^2. Draw rough sketches of the evolution with time of the initial profile $h(x, 0) = \exp(-x^2)$.

(d) If the plate is horizontal (*i.e.* $\alpha = 0$), show that an appropriate velocity scale is $U = \delta^3 \rho g L^2/\mu$ and that lubrication theory leads in this case to the thin-film equation

$$\frac{\partial h}{\partial t} = \frac{\partial}{\partial x}\left(\frac{h^3}{3}\frac{\partial h}{\partial x}\right).$$