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B5.3 Viscous Flow Lecture 1

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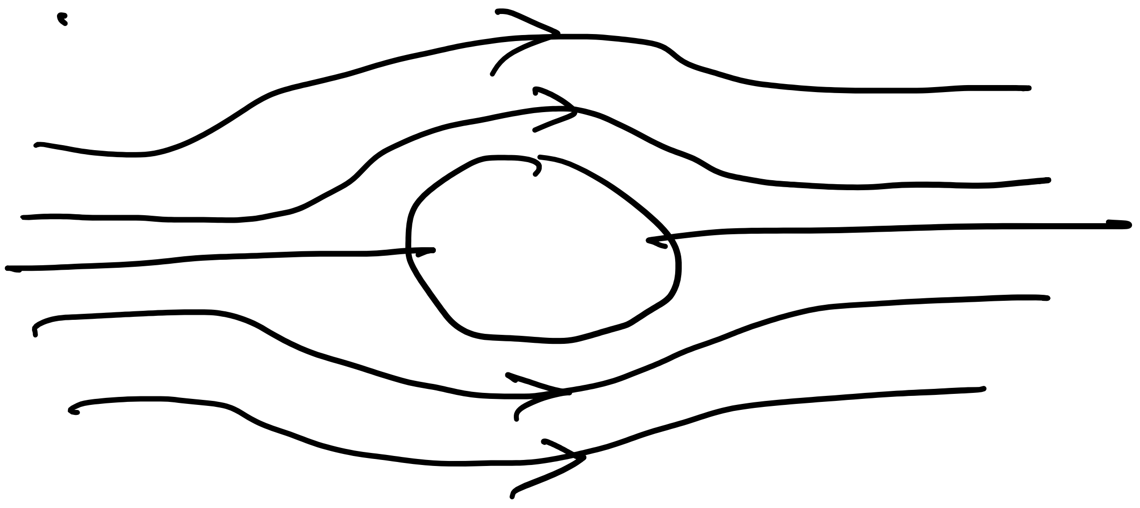
Motivation: Inviscid fluid theory (Part A) cannot explain many phenomena:

D'Alembert's paradox: flow round a cylinder is fore/aft symmetric so there's no drag:



A dusty car stays dusty when driven
A dirty window stays dirty when it rains.

"Hydraulics observes phenomena that cannot be explained. Theoretical fluid mechanics explains phenomena that cannot be observed."

Sir Cyril Hinshelwood
Tutor at Trinity (1921–1937)
Nobel Prize in Chemistry (1956)

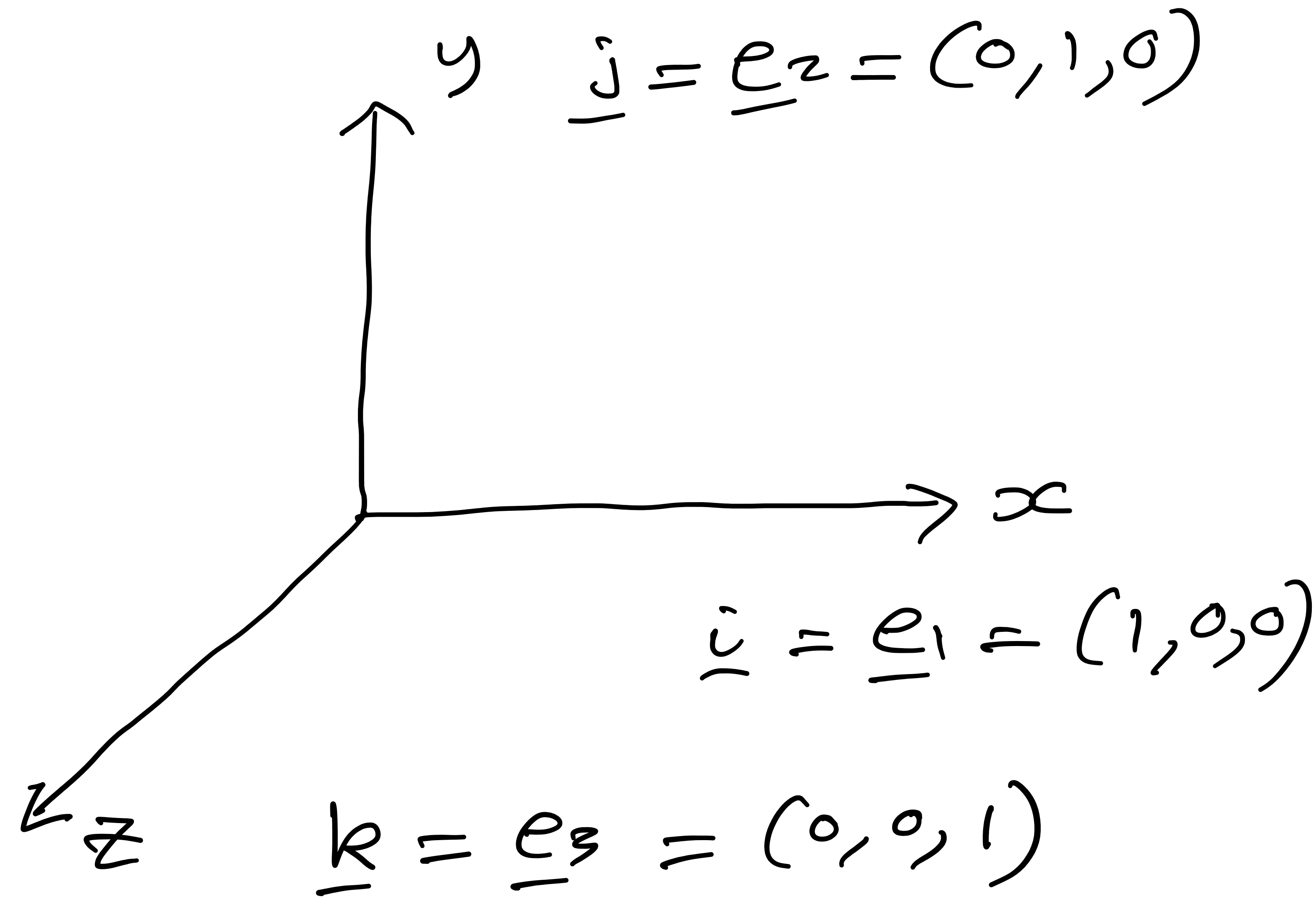
Feynman Lectures on Physics: Two chapters on fluids:
Flow of dry water (Part A Fluids)
Flow of wet water (Viscous Flow)

No formal prerequisites, but builds on and shares notation with Part A Fluids. Uses ideas about boundary layers from the last part of DEs 2.

As in Part A, we treat the fluid as a continuum with density $\rho(\underline{x},t)$, velocity $\underline{u}(\underline{x},t)$ and temperature $T(\underline{x},t)$. All are smooth functions of position $\underline{x}$ and time t.

[Smoothness relaxed in Waves and Compressible Flow]

Use Cartesian coordinates $Oxyz$ with standard orthonormal basis vectors:



Employ the summation convention:

Implicit sums over repeated pairs of indices in an expression.

Examples: $\underline{x} = (x,y,z) = x_i \underline{e}_i$

$\underline{u} = (u,v,w) = u_i \underline{e}_i$

If $f(\underline{x})$ and $\underline{G}(\underline{x})$ are differentiable:

$$\nabla f = \underline{e}_i \frac{\partial f}{\partial x_i}$$

$$\nabla \cdot \underline{G} = \frac{\partial G_i}{\partial x_i}$$

$$\nabla \wedge \underline{G} = \underline{e}_i \wedge \frac{\partial \underline{G}}{\partial x_i}$$

implied sum over i

The convective derivative

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \underline{u} \cdot \nabla = \frac{\partial}{\partial t} + u_i \frac{\partial}{\partial x_i}$$

A product with 3 or more of the same index is an invalid expression.

We can do better than this by thinking about the ith component of a vector expression.

E.g. $[\nabla \wedge \underline{G}]_i = \underline{e}_i \cdot (\nabla \wedge \underline{G})$

$$= \underline{e}_i \cdot \left(\underline{e}_j \wedge \frac{\partial \underline{G}}{\partial x_j} \right)$$

unpaired indices must match.

$$= \underline{e}_i \cdot \left(\underline{e}_j \wedge \frac{\partial}{\partial x_j} \big(\underbrace{G_k \underline{e}_k}_{= \underline{G}} \big) \right)$$

$$= \underline{e}_i \cdot (\underline{e}_j \wedge \underline{e}_k) \frac{\partial}{\partial x_j} G_k$$

$$= \varepsilon_{ijk} \frac{\partial}{\partial x_j} G_k$$

where $\varepsilon_{ijk} = \underline{e}_i \cdot (\underline{e}_j \wedge \underline{e}_k)$

$$= \begin{cases} 1 & \text{if } i,j,k \text{ are cyclic permutation of } 1,2,3 \\ -1 & \text{if } i,j,k \text{ are an anticyclic permutation of } 1,2,3 \\ 0 & \text{otherwise (repeated indices)} \end{cases}$$

$$\varepsilon_{ijk} = \varepsilon_{jki} = \varepsilon_{kij} = -\varepsilon_{kji}$$

$$\varepsilon_{ijk} \varepsilon_{pqk} = \delta_{ip}\delta_{jq} - \delta_{iq}\delta_{jp}$$

repeated index k

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$$[\underline{a} \wedge \underline{b}]_i = \varepsilon_{ijk}\, a_j\, b_k$$

$$[(\nabla \wedge \underline{u}) \wedge \underline{u}]_i = \varepsilon_{ijk}\, [\nabla \wedge \underline{u}]_j\, u_k$$

$$= \varepsilon_{ijk} \underbrace{\left(\varepsilon_{jpq} \frac{\partial}{\partial x_p} u_q\right)}_{[\nabla \wedge \underline{u}]_j} u_k$$

$$= \varepsilon_{jki}\, \varepsilon_{jpq} \left(\frac{\partial}{\partial x_p} u_q\right) u_k$$

matching index

$$= (\delta_{kp}\delta_{iq} - \delta_{kq}\delta_{ip})\, u_k \frac{\partial}{\partial x_p} u_q$$

$$= u_k \frac{\partial}{\partial x_k} u_i - u_k \frac{\partial}{\partial x_i} u_k$$

$$= u_k \frac{\partial}{\partial x_k} u_i - \frac{\partial}{\partial x_i}\left(\tfrac{1}{2} u_k u_k\right)$$

$$= \left[\underline{u} \cdot \nabla \underline{u} - \nabla\left(\tfrac{1}{2} |\underline{u}|^2\right)\right]_i$$

This is true for $i = 1, 2, 3$ so

$$(\nabla \wedge \underline{u}) \wedge \underline{u} = \underline{u} \cdot \nabla \underline{u} - \nabla\left(\tfrac{1}{2} |\underline{u}|^2\right).$$

Kinematics

(as in Part A, but examinable in Viscous Flow. See online notes for details)

Reynolds' Transport Theorem (RTT)

If $V(t)$ is a material volume advected with the fluid velocity $\underline{u}(\underline{x},t)$ and $f(\underline{x},t)$ is continuously differentiable then

$$\frac{d}{dt}\iiint_{V(t)} f(\underline{x},t)\,dV = \iiint_{V(t)} \frac{\partial f}{\partial t} + \nabla\cdot(f\underline{u})\,dV$$

$$\text{RHS} = \iiint_{V(t)} \frac{\partial f}{\partial t}\,dV + \iint_{\partial V(t)} f\,\underline{u}\cdot\underline{n}\,dS$$

by divergence thm

$$= \text{contribution from } f \text{ changing} + \text{contribution from boundary } \partial V(t) \text{ moving with velocity } \underline{u}(\underline{x},t)$$

Mass conservation

A material volume always comprises the same fluid elements, by definition, so its mass cannot change:

$$0 = \frac{d}{dt}\iiint_{V(t)} \rho(\underline{x},t)\,dV$$

$$= \iiint_{V(t)} \frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\underline{u})\,dV$$

by RTT with $f=\rho$.

This holds for all material volumes $V(t)$ so the integrand, assuming its continuous, must vanish pointwise:

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\underline{u}) = 0.$$

Continuity equation (aka mass conservation)

If the integrand were >0 at $\underline{x}_0$ say, by continuity the integrand >0 in some small $V(t)$ containing $\underline{x}_0$, so the integral >0. Contradiction

The continuity equation is equivalent to $\frac{D\rho}{Dt} + \rho\nabla\cdot\underline{u} = 0$,

where $\frac{D}{Dt} = \frac{\partial}{\partial t} + \underline{u}\cdot\nabla = \frac{\partial}{\partial t} + u_i\frac{\partial}{\partial x_i}$ is the material or Lagrangian time derivative.

For an incompressible fluid $\frac{D\rho}{Dt}=0$, so $\nabla\cdot\underline{u}=0$.