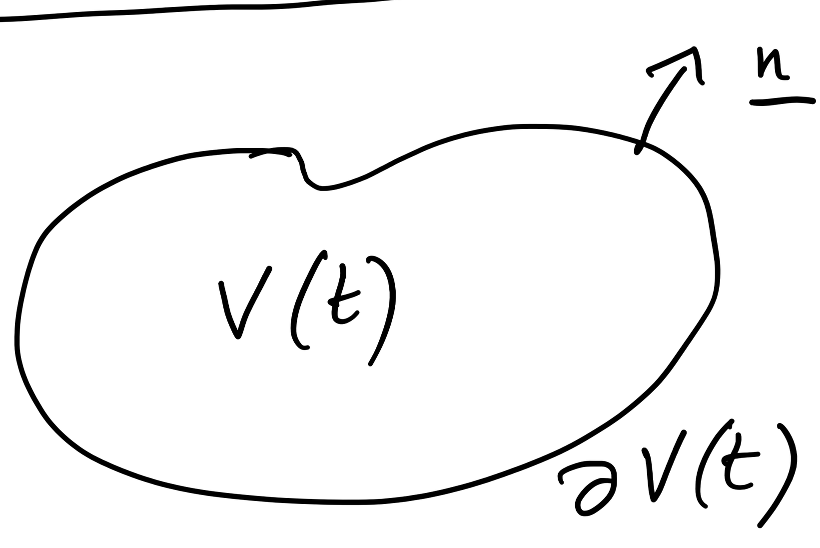
Sunday, 11 October 2020 18:28

Viscous Flow lecture 2

Last time: viscous vs inviscid fluids
(wet) vs (dry)
Reynolds' Transport Theorem (RTT)
Mass conservation

Dynamics – momentum conservation

Apply Newton's 2nd Law (NII) to a material volume $V(t)$



Body forces like gravity acting on each piece of fluid from outside

Internal forces that are transmitted locally across surfaces.

$$(\text{NII}) \quad \frac{d}{dt}\iiint_{V(t)} \rho \underline{u}\, dV$$ rate of change of linear momentum inside $V(t)$

no dot

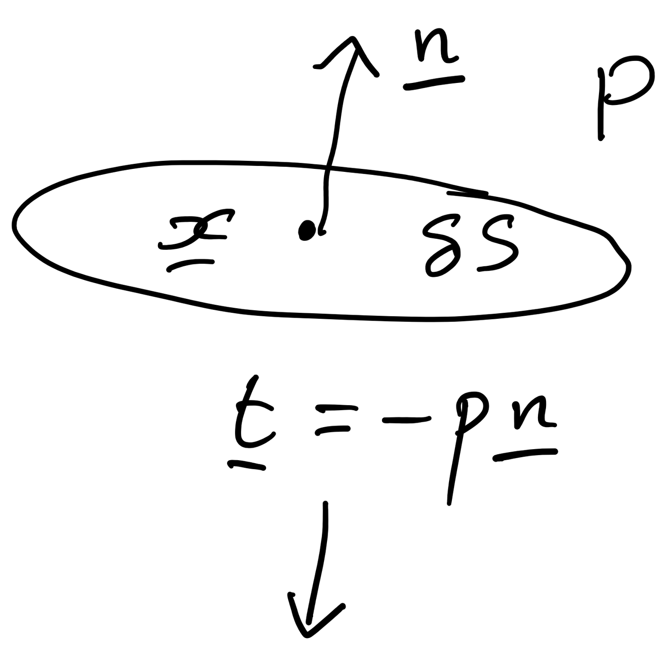
$$= \iint_{\partial V(t)} \underline{t}(\underline{n})\, dS + \iiint_{V(t)} \rho \underline{F}\, dV$$

net force at surface (internal) — net body force on volume (external forces)

$\underline{F}(\underline{x},t)$ is the external body force per unit, e.g. gravity when $\underline{F}(\underline{x},t) = \underline{g}$.
$\rho \underline{F}$ is the body force per unit volume, so the total body force exerted is
$$\iiint_{V(t)} \rho \underline{F}\, dV.$$

The stress vector $\underline{t}(\underline{x},t,\underline{n})$ is the force per unit area (stress) exerted on the surface element at $\underline{x}$ by the fluid towards which the normal $\underline{n}$ points.
$\underline{t}$ is sometimes called the traction vector, hence $\underline{t}$.

For an inviscid fluid $\underline{t} = -p\,\underline{n}$, with no tangential component.



$p>0$ push downwards on the surface δS.

Applying RTT with $f = \rho u_i$ to NII gives
$$\iiint_{V(t)} \frac{\partial}{\partial t}(\rho u_i) + \nabla\cdot(\rho \underline{u} u_i)\, dV$$
$$= \iint_{\partial V(t)} -p n_i\, dS + \iiint_{V(t)} \rho F_i\, dV$$

$$\iiint_{V(t)} \rho\left(\frac{\partial u_i}{\partial t} + \underline{u}\cdot\nabla u_i\right) + u_i \underbrace{\left(\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\underline{u})\right)}_{=0 \text{ by mass conservation (corollary to RTT)}} dV$$

$$= \iiint_{V(t)} -\frac{\partial p}{\partial x_i} + \rho F_i\, dV$$ by divergence thm

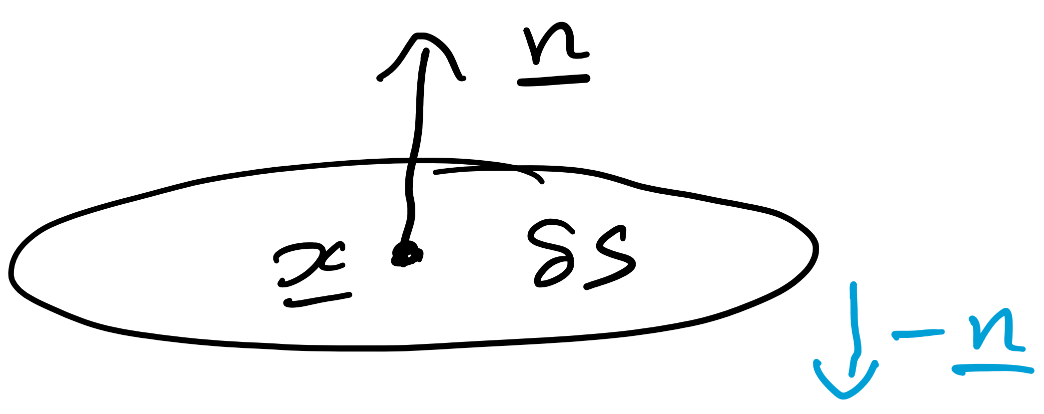
True for all material volumes $V(t)$ so
$$\rho \frac{D\underline{u}}{Dt} = -\nabla p + \rho \underline{F}$$
Euler momentum equation for an inviscid fluid.

Applying the divergence theorem to
$$\iint_{\partial V(t)} -p\,\underline{n}\, dS$$
which is a vector-valued expression.
Dot with an arbitrary constant vector $\underline{c}$,
$$\underline{c}\cdot \iint_{\partial V(t)} -p\underline{n}\, dS = \iint_{\partial V(t)} (-p\underline{c})\cdot\underline{n}\, dS$$
$$= \iiint_{V(t)} \nabla\cdot(-p\underline{c})\, dV \quad \text{by div thm}$$
$$= \underline{c}\cdot\iiint_{V(t)} -\nabla p\, dV$$
True for all $\underline{c}$ so
$$\iint_{\partial V(t)} -p\underline{n}\, dS = \iiint_{V(t)} -\nabla p\, dV.$$

How can we generalize this, writing
$$\iint_{\partial V(t)} \underline{t}(\underline{x},t,\underline{n})\, dS$$ as a volume integral for general $\underline{t}$?

Sunday, 11 October 2020 18:28

Newton's 3rd Law (NIII) action & reaction



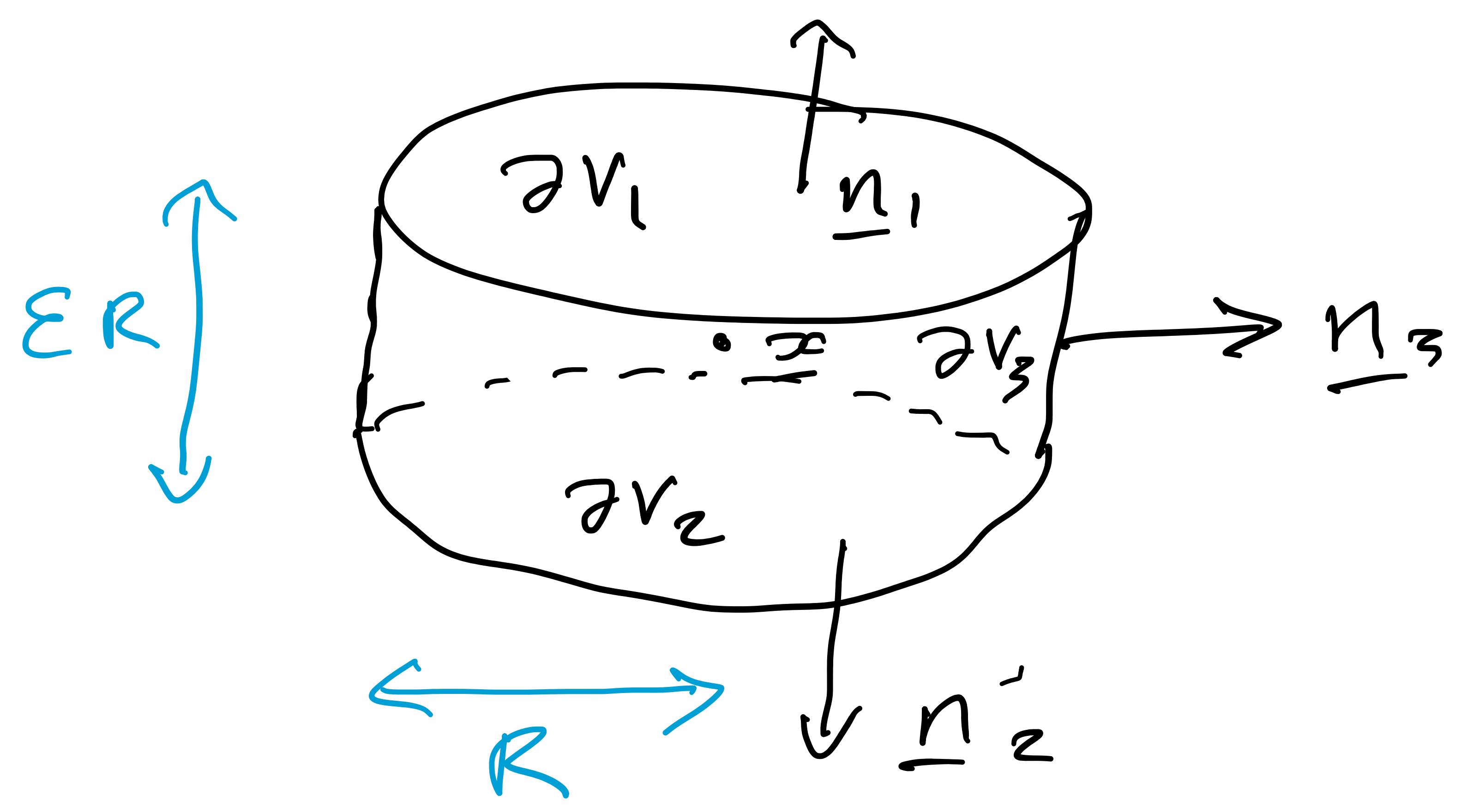
The force exerted by the upper side (to which $\underline{n}$ points) is $\underline{t}(\underline{n})\,\delta S$.

The force exerted by the lower side (to which $-\underline{n}$ points) is $\underline{t}(-\underline{n})\,\delta S$.

$$\text{NIII} \Rightarrow \underline{t}(\underline{n}) = -\underline{t}(-\underline{n})$$

Why is this?

Consider a material volume $V(t)$ that at time t comprises a cylinder of radius R and thickness εR, centre $\underline{x}$.



$$\text{NII}: \iiint_{V(t)} \rho \frac{D\underline{u}}{Dt} - \rho \underline{F}\, dV = \iint_{\partial V(t)} \underline{t}(\underline{x}, t, \underline{n})\, dS$$

The LHS is $O(\varepsilon R^3)$ by the integral mean value theorem:

$$\left| \iiint_{V(t)} \underline{A}(\underline{x})\, dV \right| \leq \text{vol}(V(t)) \sup_{\underline{x} \in V(t)} |\underline{A}(\underline{x})|$$

$$\text{vol}(V(t)) = \pi \varepsilon R^3$$

As $\varepsilon \to 0$, $R \to 0$

$$\begin{aligned} \text{RHS} = \ & \pi R^2\, \underline{t}(\underline{x}, t, \underline{n}_1) \\ & + \pi R^2\, \underline{t}(\underline{x}, t, \underline{n}_2) \\ & + o(\varepsilon R^2), \end{aligned}$$

so

$$O(\varepsilon R^3) = \pi R^2 \left(\underline{t}(\underline{x}, t, \underline{n}) + \underline{t}(\underline{x}, t, -\underline{n}) \right) + o(\varepsilon R^2)$$

since $\underline{n}_1 = \underline{n}$ and $\underline{n}_2 = -\underline{n}$.

Can only hold as $\varepsilon \to 0$ if

$$\underline{t}(\underline{x}, t, \underline{n}) = -\underline{t}(\underline{x}, t, -\underline{n}).$$

Loosely, if $\underline{t}(\underline{x}, t, \underline{n}) \neq -\underline{t}(\underline{x}, t, -\underline{n})$ we would have a finite net force on a surface which has zero thickness, and hence has zero volume and zero mass. This would cause an infinite acceleration of the surface.

→ suggests $\underline{t}$ might be linear in $\underline{n}$.

Sunday, 11 October 2020 18:28

Linear relations between vectors such as $\underline{t}$ and $\underline{n}$ are mediated by tensors.

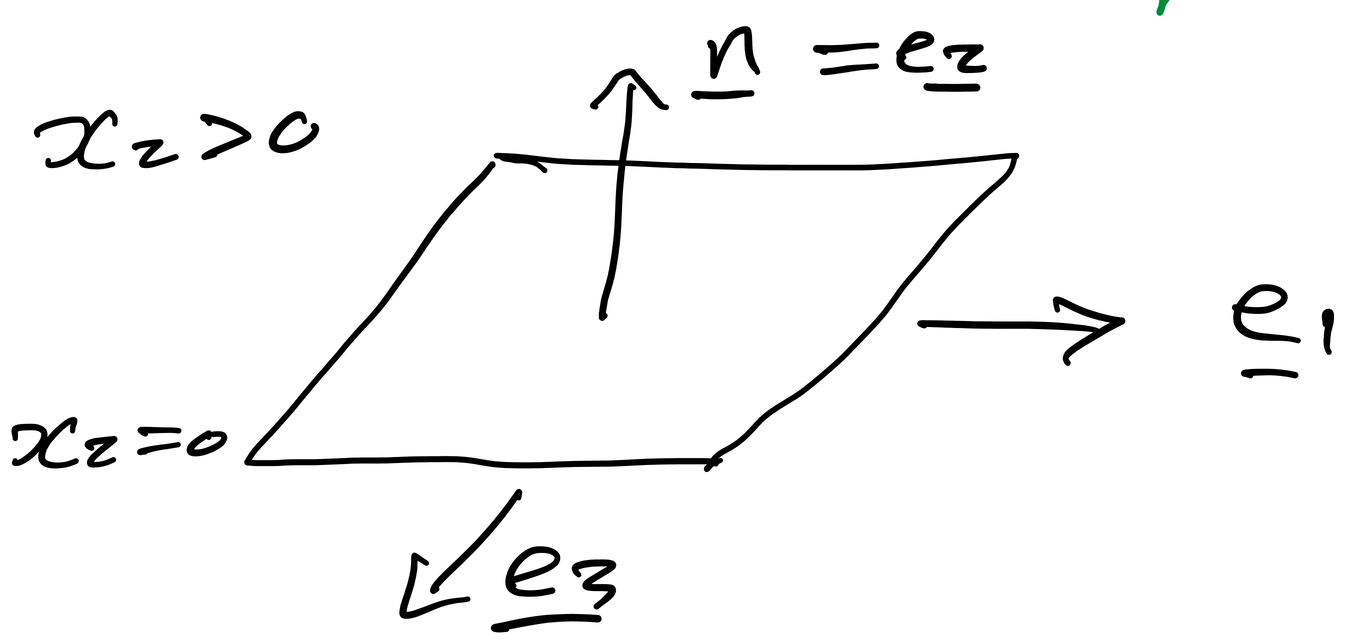
The stress tensor with components σ_{ij} is defined by σ_{ij} being the component of stress in the $\underline{e}_i$-direction exerted on a surface element with normal $\underline{e}_j$ by the fluid towards which $\underline{e}_j$ points,

i.e. $\sigma_{ij} = \underline{e}_i \cdot \underline{t}(\underline{e}_j)$

or $\underline{t}(\underline{e}_j) = \underline{e}_i \sigma_{ij}$ (summation over i)

Not universal which index is which on σ, but we'll find shortly that $\sigma_{ij} = \sigma_{ji}$ so it doesn't matter.

E.g. the stress exerted by fluid in $x_2 > 0$ on a plate at $x_2 = 0$ is $\underline{t}(\underline{e}_2) = \underbrace{\underline{e}_1 \sigma_{12} + \underline{e}_3 \sigma_{32}}_{\text{tangential stress}} + \underbrace{\underline{e}_2 \sigma_{22}}_{\text{normal stress}}$



For an inviscid fluid $\underline{t}(\underline{e}_j) = -p\underline{e}_j$

so $\sigma_{ij} = \underline{e}_i \cdot (-p\,\underline{e}_j) = -p\,\delta_{ij}$

is purely normal