

Viscous Flow lecture 3

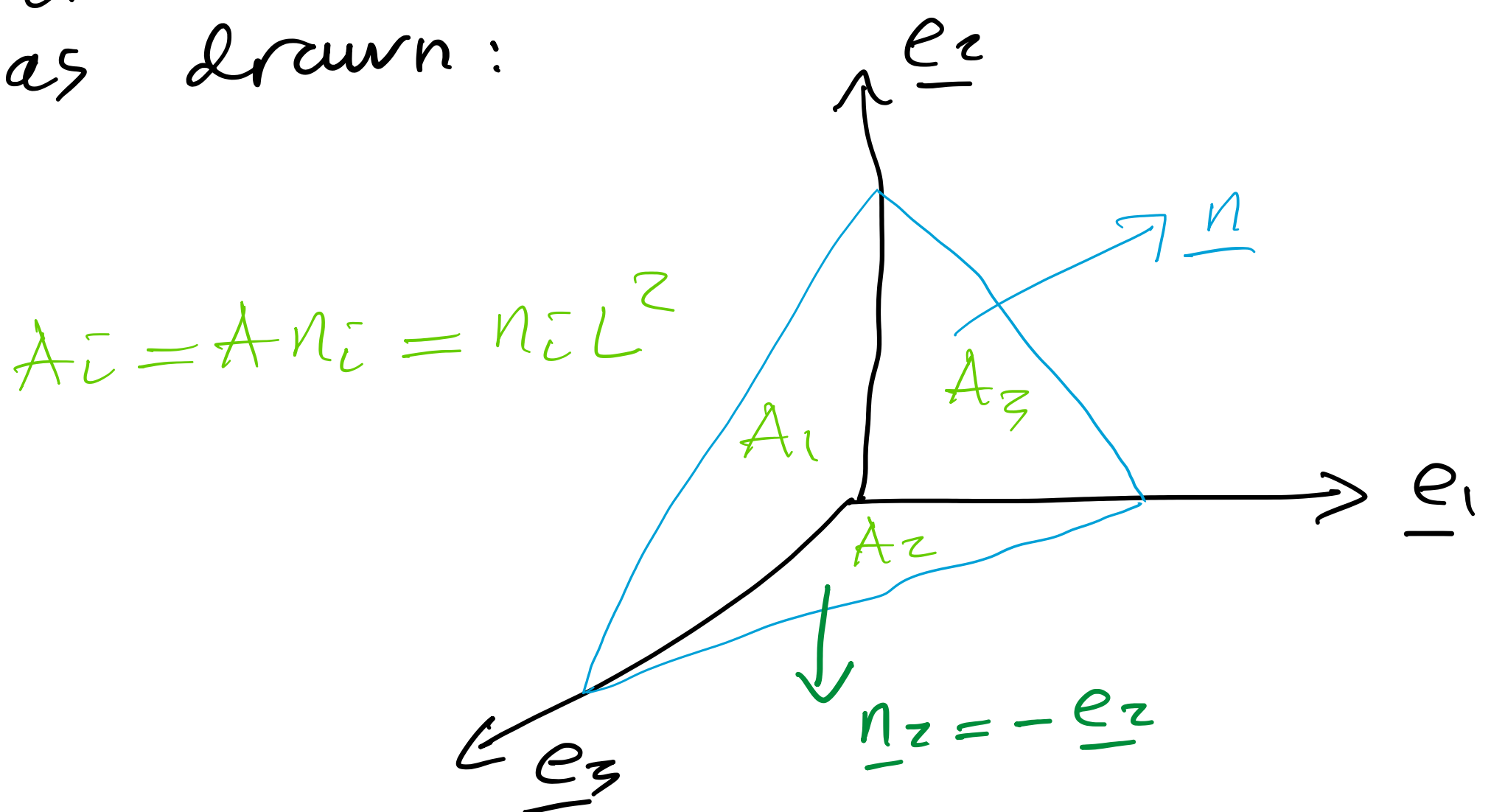
Last time: Newton's 3rd Law

$$\underline{t}(\underline{n}) = -\underline{t}(-\underline{n})$$

Cauchy's Stress Theorem

$$\underline{t}(\underline{n}) = \underline{e}_i \sigma_{ij} n_j$$

Proof: Consider a material volume $V(t)$ that is instantaneously a tetrahedron as drawn:



The sloping face has area $A = L^2$, defining a length scale L , and outward normal $\underline{n}$.

The three faces A_i in the planes $x_i = 0$ have areas $A_i = A n_i = n_i L^2$.

$$NII \Rightarrow \iiint_{V(t)} \rho \frac{D\underline{u}}{Dt} - \rho \underline{F} dV = \iint_{\partial V(t)} \underline{t}(\underline{n}) dS$$

LHS is $O(L^3)$ as $L \rightarrow 0$, assuming the integrand is continuous, hence bounded, in $V(t)$.

$$RHS = \underline{t}(\underline{n}) L^2 + \underline{t}(-\underline{e}_j) n_j L^2 + O(L^3)$$

where $\underline{t}$ is evaluated at $\underline{x} = 0$ to sufficient accuracy.

Taking $L \rightarrow 0$ establishes that

$$\underline{t}(\underline{n}) + \underline{t}(-\underline{e}_j) n_j = 0.$$

$$NIII \Rightarrow \underline{t}(-\underline{e}_j) = -\underline{t}(\underline{e}_j) = -\underline{e}_i \sigma_{ij}$$

$$\therefore \underline{t}(\underline{n}) - \underline{e}_i \sigma_{ij} n_j = 0$$

This is Cauchy's stress theorem.

Again, the net force on an arbitrarily small volume must vanish to avoid an infinite acceleration.

Given 9 quantities σ_{ij} we can compute

$$\underline{t}(\underline{n}) = \underline{e}_i \sigma_{ij} n_j$$

for any direction $\underline{n}$ of the normal.

Now we can convert that surface stress integral into a volume integral.

$$\begin{aligned} \iint_{\partial V(t)} \underline{t}(\underline{n}) dS &= \underline{e}_i \iint_{\partial V(t)} \sigma_{ij} n_j dS \\ &= \underline{e}_i \iiint_{V(t)} \frac{\partial}{\partial x_j} \sigma_{ij} dV \end{aligned}$$

by the divergence theorem.

is why some people define this as

$$\frac{\partial}{\partial x_j} \sigma_{ji}$$

NII for $V(t)$ becomes

$$\iiint_{V(t)} \rho \frac{D\underline{u}}{Dt} - \underline{e}_i \frac{\partial \sigma_{ij}}{\partial x_j} - \rho \underline{F} dV = 0$$

True for all material volumes $V(t)$ so

$$\rho \frac{D\underline{u}}{Dt} = \underline{e}_i \frac{\partial \sigma_{ij}}{\partial x_j} + \rho \underline{F}$$

For inviscid fluids, $\sigma_{ij} = -p \delta_{ij}$ so we recover the Euler equation

$$\rho \frac{D\underline{u}}{Dt} = -\nabla p + \rho \underline{F}$$

$$\iint_{\partial V(t)} \underline{t}(\underline{n}) dS$$

Looks like it should
be proportional to
the area $|\partial V(t)| = O(L^2)$

We've shown that

$$\iint_{\partial V(t)} \underline{t}(\underline{n}) dS = \iiint_{V(t)} \underline{e}_i \frac{\partial \sigma_{ij}}{\partial x_j} dV$$

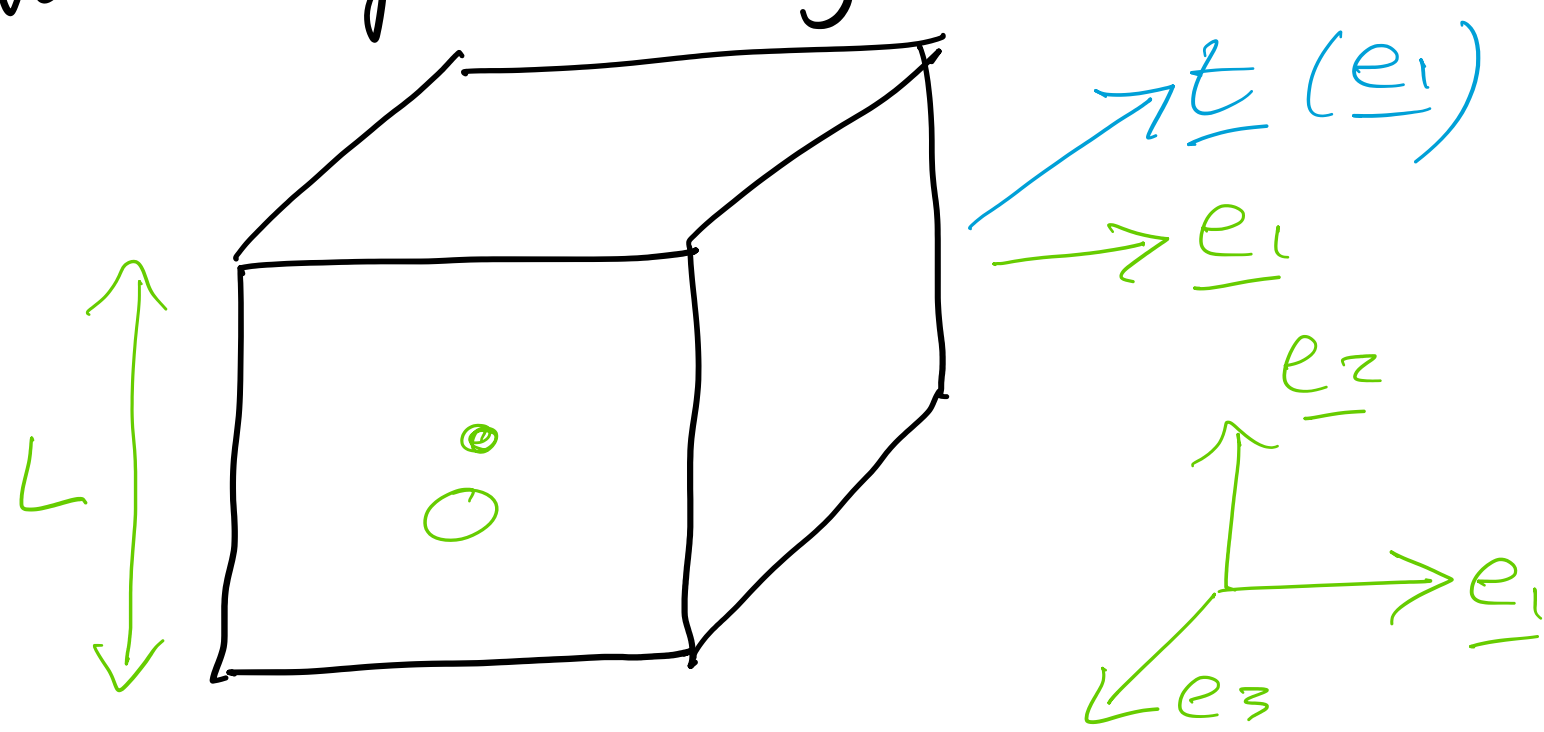
is in fact proportional to the
volume L^3 .

If σ_{ij} is constant, the integral
vanishes — the flux in exactly
balances the flux out.

Properties of σ_{ij} in general

① It is symmetric, $\sigma_{ij} = \sigma_{ji}$

Proof: Consider a small material volume $V(t)$ instantaneously forming a cube centred at O with faces $x_j = \pm \frac{1}{2} L$.



Conservation of angular momentum:

$$\frac{d}{dt} \iiint_{V(t)} \underline{x} \wedge (\rho \underline{u}) dV = \iint_{\partial V(t)} \underline{x} \wedge \underline{t}(\underline{n}) dS + \iiint_{V(t)} \underline{x} \wedge \rho \underline{F} dV$$

Applying RTT with $f = \underline{e}_i \cdot (\underline{x} \wedge \rho \underline{u})$ gives

$$\iiint_{V(t)} \underline{x} \wedge \left(\rho \frac{D\underline{u}}{Dt} - \rho \underline{F} \right) dV = \iint_{\partial V(t)} \underline{x} \wedge \underline{t}(\underline{n}) dS$$

$|\underline{x}| = O(L)$ so LHS is $O(L^4)$

$$\begin{aligned} \text{RHS} &= \frac{L}{2} \underline{e}_j \wedge L^2 \underline{t}(\underline{e}_j) \\ &\quad + \left(-\frac{L}{2} \underline{e}_j \right) \wedge L^2 \underline{t}(-\underline{e}_j) \\ &\quad + O(L^4) \end{aligned}$$

Again evaluating $\underline{t}$ at the origin O to sufficient accuracy.

$$\text{NII} \Rightarrow \underline{t}(-\underline{e}_j) = -\underline{t}(\underline{e}_j)$$

$$\text{so RHS} = L^3 \underline{e}_j \wedge \underline{t}(\underline{e}_j) + O(L^4)$$

$$\text{LHS} = O(L^4).$$

This has to be true for arbitrarily small L , so $\underline{e}_j \wedge \underline{t}(\underline{e}_j) = 0$

We know $\underline{t}(\underline{e}_j) = \underline{e}_i \sigma_{ij}$ by Cauchy's Stress Theorem so

$$\underline{e}_j \wedge \underline{e}_i \sigma_{ij} = 0.$$

$$\begin{aligned} \underline{e}_1 (\sigma_{32} - \sigma_{23}) + \underline{e}_2 (\sigma_{13} - \sigma_{31}) \\ + \underline{e}_3 (\sigma_{21} - \sigma_{12}) = 0 \end{aligned}$$

$\therefore \sigma_{ij} = \sigma_{ji}$ is symmetric.

Alternatively: $\underline{e}_j \wedge \underline{e}_i$ is antisymmetric in swapping i & j

$$\text{i.e. } \underline{e}_j \wedge \underline{e}_i = -\underline{e}_i \wedge \underline{e}_j$$

$\therefore \sigma_{ij}$ must be symmetric to make

$$\underbrace{\underline{e}_j \wedge \underline{e}_i}_{\text{anti-symmetric}} \underbrace{\sigma_{ij}}_{\text{symmetric}} = 0$$

σ_{ij} are the components of a tensor

Proof: σ_{ij} relates two vectors $\underline{t} \in \mathbb{R}^n$
via $\underline{t}(\underline{n}) = \underline{e}_i \sigma_{ij} n_j$

In components: $t_i = \sigma_{ij} n_j$

Consider a rotation from axes
 $Ox_1x_2x_3$ with basis $\underline{e}_1, \underline{e}_2, \underline{e}_3$
to $Ox_1'x_2'x_3'$ with basis $\underline{e}_1', \underline{e}_2', \underline{e}_3'$.

For any vector, $\underline{r} = x_j \underline{e}_j = x_i' \underline{e}_i'$
 $x_i' = \underline{e}_i' \cdot \underline{r} = \underline{e}_i' \cdot \underline{e}_j x_j = L_{ij} x_j$

$L_{ij} = \underline{e}_i' \cdot \underline{e}_j$ are the components
of an orthogonal matrix L , so $LL^T = I$.

$$\therefore x_j = (L^T)_{ji} x_i' = L_{ij} x_i'$$

Applying these transformations to
Cauchy's stress theorem:

$$t_i' = L_{ij} t_j = L_{ij} \sigma_{jk} n_k \quad (1)$$

$$\text{and } t_i' = \sigma_{ij}' n_j' = \sigma_{ij}' L_{jk} n_k \quad (2)$$

Comparing (1) & (2) for all n_k

$$\Rightarrow L_{ij} \sigma_{jk} = \sigma_{ij}' L_{jk}$$

In matrix notation

$$L \sigma = \sigma' L$$

Multiply on the right by L^T to
get

$$L \sigma L^T = \sigma'$$

so σ transforms like a (rank 2)
tensor, compare with

$$x_i' = L_{ij} x_j$$

$$\underline{x}' = L \underline{x}$$

Think about how the tensors
 $t_i t_j$ or $n_i n_j$ transform
given we know how the vectors
 t_i and n_i transform.