Viscous Flow lecture 5

last time: we derived the Newtonian constitutive relation

$$\tau_{ij} = \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + \lambda\, \delta_{ij} \underbrace{\frac{\partial u_k}{\partial x_k}}_{=\nabla\cdot\underline{u}}$$

shear viscosity μ
bulk viscosity λ

$$\sigma_{ij} = -p\delta_{ij} + \tau_{ij}$$

For incompressible fluids ($\nabla\cdot\underline{u}=0$) we get

$$\tau_{ij} = \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

If we also assume that μ is constant (not a function of p, ρ, T) then

$$\frac{\partial \sigma_{ij}}{\partial x_j} = \frac{\partial}{\partial x_j}\left(-p\delta_{ij} + \mu\frac{\partial u_i}{\partial x_j} + \mu\frac{\partial u_j}{\partial x_i}\right)$$

$$= -\frac{\partial p}{\partial x_i} + \mu\frac{\partial^2 u_i}{\partial x_j \partial x_j} + \mu\underbrace{\frac{\partial^2 u_j}{\partial x_j \partial x_i}}$$

$$= -\frac{\partial p}{\partial x_i} + \mu\nabla^2 u_i \qquad = \frac{\partial}{\partial x_i}\frac{\partial u_j}{\partial x_j} = 0$$

Cauchy's momentum equation

$$\rho\frac{D\underline{u}}{Dt} = \underline{e_i}\frac{\partial \sigma_{ij}}{\partial x_j} + \rho\underline{F}$$

becomes (with constant density ρ)

(NS2) $$\rho\frac{D\underline{u}}{Dt} = -\nabla p + \underbrace{\mu\nabla^2\underline{u}}_{\text{new term}} + \rho\underline{F}$$

(NS1) $$\nabla\cdot\underline{u} = 0.$$

These are the incompressible Navier–Stokes equations.

These are 4 equations (3+1) for p, u_1, u_2, u_3, 4 unknowns.

$\nabla\cdot$(NS2) gives

$$\rho\left(\nabla\cdot\frac{\partial\underline{u}}{\partial t} + \nabla\cdot(\underline{u}\cdot\nabla\underline{u})\right) = -\nabla^2 p + \mu\nabla\cdot(\nabla^2\underline{u}) + \rho\nabla\cdot\underline{F}$$

$$\rho\left(\frac{\partial}{\partial t}\cancelto{0}{\nabla\cdot\underline{u}} + \nabla\cdot(\underline{u}\cdot\nabla\underline{u})\right) = -\nabla^2 p + \cancelto{0}{\mu\nabla^2\nabla\cdot\underline{u}} + \rho\nabla\cdot\underline{F}$$

This gives Poisson's equation

$$\nabla^2 p = \rho\nabla\cdot\underline{F} - \underbrace{\nabla\cdot(\underline{u}\cdot\nabla\underline{u})}_{=\frac{\partial}{\partial x_j}\left(u_i\frac{\partial u_j}{\partial x_i}\right)}$$

For non-Cartesian coordinates it's convenient to use the vector identities

$$\underline{u}\cdot\nabla\underline{u} = (\nabla\wedge\underline{u})\wedge\underline{u} + \nabla\left(\tfrac{1}{2}|\underline{u}|^2\right)$$

$$\nabla^2\underline{u} = \nabla(\nabla\cdot\underline{u}) - \nabla\wedge(\nabla\wedge\underline{u})$$

to rewrite (NS2) as

(NS2')
$$\frac{\partial\underline{u}}{\partial t} + \underline{\omega}\wedge\underline{u} + \nabla\left(\frac{p}{\rho} + \tfrac{1}{2}|\underline{u}|^2\right) = -\nu\nabla\wedge\underline{\omega} + \underline{F}$$

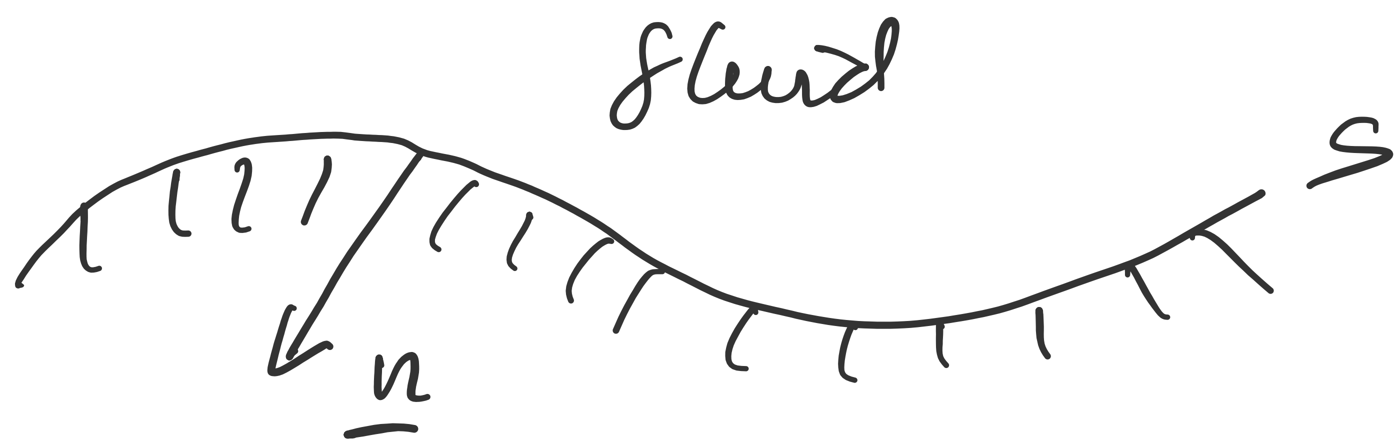
where $\nu = \mu/\rho$ is the kinematic viscosity, units of $m^2 s^{-1}$ like a diffusivity, and $\underline{\omega} = \nabla\wedge\underline{u}$ is the vorticity.

The Navier–Stokes equations involve second spatial derivatives of $\underline{u}$, as either $\nabla^2\underline{u}$ or $\nabla\wedge\underline{\omega}$ so we need more boundary conditions.

The Euler equations only require first spatial derivatives of $\underline{u}$.

Boundary conditions

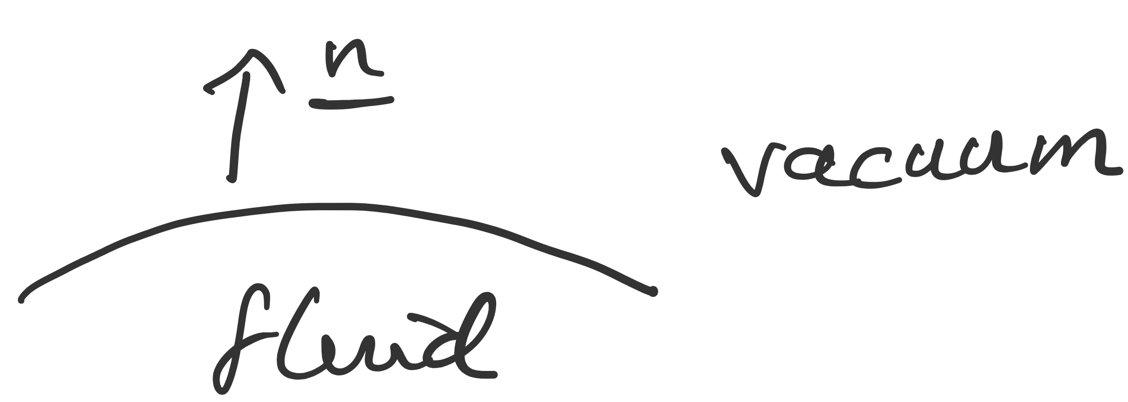
1) Rigid impermeable boundary S moving with velocity $\underline{U}$.



i) no-flux BC : $\underline{u} \cdot \underline{n} = \underline{U} \cdot \underline{n}$ on S
(same as for Euler)

ii) no-slip BC : $\underline{u} \wedge \underline{n} = \underline{U} \wedge \underline{n}$ on S
New, now the tangential velocity must also be continuous.
Together $\Rightarrow$ $\underline{u} = \underline{U}$ on S

2) Free surface Γ moving with outward normal velocity V



i) no-flux BC : $\underline{u} \cdot \underline{n} = V$ on Γ
ii) no-stress BC : $\underline{t}(\underline{n}) = 0$ on Γ

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Vorticity

$\underline{\omega} = \nabla \wedge \underline{u}$ measures the local rotation of fluid elements.

Imagine floating little twigs on the fluid surface & watching them rotate.

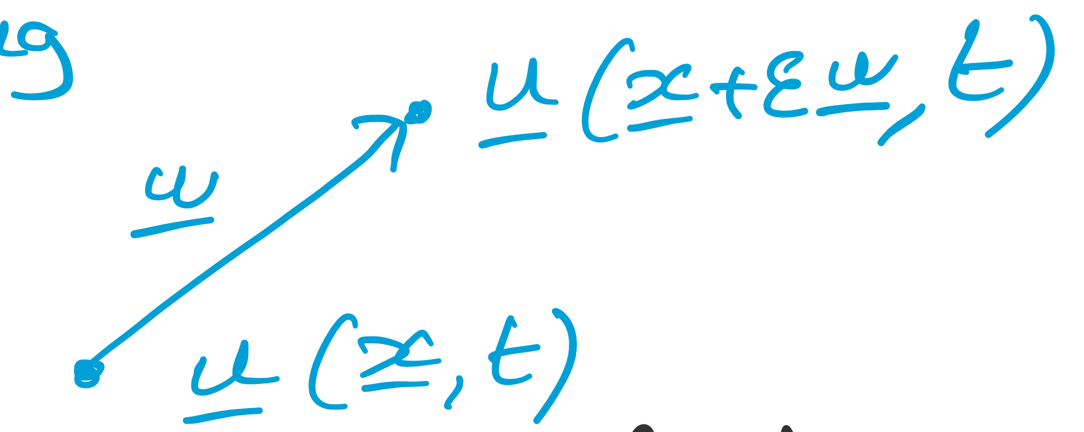
If $\underline{F}$ is conservative $(\nabla \wedge \underline{F} = 0)$ then $\nabla \wedge$ (NS2') gives (VTE)

$$\frac{\partial \underline{\omega}}{\partial t} + \underline{u} \cdot \nabla \underline{\omega} - \underline{\omega} \cdot \nabla \underline{u} = \nu \nabla^2 \underline{\omega}$$

using $\nabla \cdot \underline{u} = 0$ and $\nabla \cdot \underline{\omega} = 0$.

This is the vorticity transport equation

$$\underbrace{\frac{D\underline{\omega}}{Dt}}_{\text{material time derivative}} - \underbrace{\underline{\omega} \cdot \nabla \underline{u}}_{\text{vortex stretching}} = \underbrace{\nu \nabla^2 \underline{\omega}}_{\text{diffusion}}$$



Vorticity can be stretched by velocity gradients, and diffuses with diffusivity ν, the kinematic viscosity.

In 2D, $\underline{u} = u(x,y,t)\underline{i} + v(x,y,t)\underline{j}$

$\underline{\omega} = \nabla \wedge \underline{u} = \omega(x,y,t)\underline{k}$

where $\omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$

The vortex stretching term $\underline{\omega} \cdot \nabla \underline{u} \equiv 0$, leaving

$$\frac{D\omega}{Dt} = \frac{\partial \omega}{\partial t} + \underline{u} \cdot \nabla \omega = \nu \nabla^2 \omega.$$

If $\nu = 0$ (Euler equations)

$\frac{D\omega}{Dt} = 0$ so $\omega \equiv 0$ initially, ω remains zero everywhere.

while $\nabla^2 \omega = 0$ inside the fluid if $\omega = 0$ everywhere, but now we can generate vorticity at boundaries.

Write 2DVTE as

$$\frac{\partial \omega}{\partial t} + \nabla \cdot (\underbrace{\underline{u}\omega - \nu \nabla \omega}_{\underline{Q}}) = 0$$

ω obeys a conservation law with flux $\underline{Q} = \underline{u}\omega - \nu \nabla \omega$.

while $\underline{u} = 0$ on a fixed rigid boundary, $\frac{\partial \omega}{\partial n} \neq 0$ in general, so $\underline{Q} \cdot \underline{n} \neq 0$ at boundaries, so we can have sources or sinks of vorticity at rigid boundaries.

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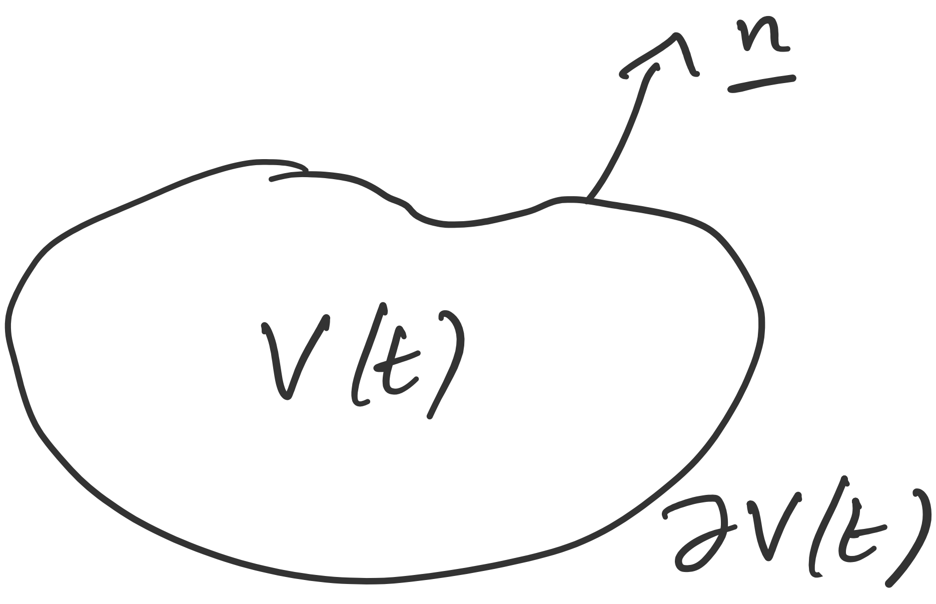
Conservation of energy

The total energy inside a material volume $V(t)$ is



$$E(t) = \iiint_{V(t)} \underbrace{\rho c_v T}_{\text{heat energy density}} + \underbrace{\tfrac{1}{2}\rho|\underline{u}|^2}_{\text{kinetic energy density}} \, dV$$

where c_v is the specific heat at constant volume (units of $J\,kg^{-1}\,K^{-1}$) and T is the temperature in Kelvin (K)

$\rho c_v T$ is the kinetic energy in the fluid molecules in addition to the kinetic energy in $\frac{1}{2}\rho|\underline{u}|^2$.

Neglecting external energy sources like radiation or chemical reactions (e.g. combustion)

$$\frac{dE}{dt} = \underbrace{\iint_{\partial V(t)} \underline{q}\cdot(-\underline{n})\,dS}_{(i)} + \underbrace{\iint_{\partial V(t)} \underline{t}(\underline{n})\cdot\underline{u}\,dS}_{(ii)}$$

$$+ \underbrace{\iiint_{V(t)} \rho \underline{F}\cdot\underline{u}\,dV}_{(iii)}$$

(i) is due to conduction of heat into V, hence $-\underline{n}$. The heat flux vector $\underline{q} = -k\nabla T$ according to Fourier's Law. The thermal conductivity k has units of $J\,m^{-1}\,s^{-1}\,K^{-1}$ to convert ∇T into an energy flux.

(ii) is rate of working of the surface stresses $\underline{t}(\underline{n})$ against $\underline{u}$.

(iii) is rate of working of body forces throughout the volume.

Using the mass and momentum conservation equations to eliminate $\frac{\partial \rho}{\partial t}$ and $\frac{\partial \underline{u}}{\partial t}$ we get

$$\rho c_v \frac{DT}{Dt} = k\nabla^2 T + \Phi$$

for ρ, c_v, k all constant.

The viscous heating (or dissipation) is

$$\Phi = \frac{1}{2}\mu \sum_{i,j}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)^2 \geq 0$$

Fluid deformation (as distinct from rigid body motions) raises the temperature.

$$\frac{DT}{Dt} = \underbrace{\frac{k}{\rho c_v}}_{=\kappa \text{ thermal diffusivity}} \nabla^2 T + \frac{1}{\rho c_v}\Phi$$

When $\underline{u} = 0$ we get

$$\frac{\partial T}{\partial t} = \kappa \nabla^2 T$$, a diffusion equation for T.