Viscous Flow lecture 6

Last time :

$$\nabla \cdot \underline{u} = 0 \qquad \text{(NS1)}$$

$$\rho \frac{D\underline{u}}{Dt} = -\nabla p + \mu \nabla^2 \underline{u} + \rho \underline{F} \qquad \text{(NS2)}$$

Unidirectional flows

Almost all explicit solutions of the unforced Navier-Stokes equations are for unidirectional flows, sometimes called shear flows.

Consider $\underline{u} = u(x, y, z, t)\, \underline{i}$

(NS1) $\Rightarrow \dfrac{\partial u}{\partial x} = 0 \Rightarrow u = u(y, z, t)$

This flow geometry $\Rightarrow \underline{u} \cdot \nabla \underline{u} \equiv 0$

(NS2y, z) $\Rightarrow \dfrac{\partial p}{\partial y} = \dfrac{\partial p}{\partial z} = 0 \Rightarrow p = p(x, t)$

(NS2x) $\Rightarrow \underbrace{\rho \dfrac{\partial u}{\partial t} - \mu \left(\dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2} \right)}_{\text{independent of } x} = \underbrace{-\dfrac{\partial p}{\partial x}}_{\text{independent of } y \,\&\, z}$

Both sides must be a function of time only, say $-G(t)$.

Hence u satisfies a 2D diffusion equation

$$\frac{\partial u}{\partial t} = \nu \left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) - \frac{G(t)}{\rho}$$

where $\nu = \mu / \rho$ is the diffusivity (units of $m^2 s^{-1}$) and $G(t)$ is called the applied pressure gradient, which must be prescribed, either explicitly or by boundary conditions. We can then solve for $u(y, z, t)$.

Typically $G(t)$ is either a constant, or sinusoidal in time.

Can solve 1D flows, steady or unsteady, and 2D steady flows using Prelims PDEs techniques.

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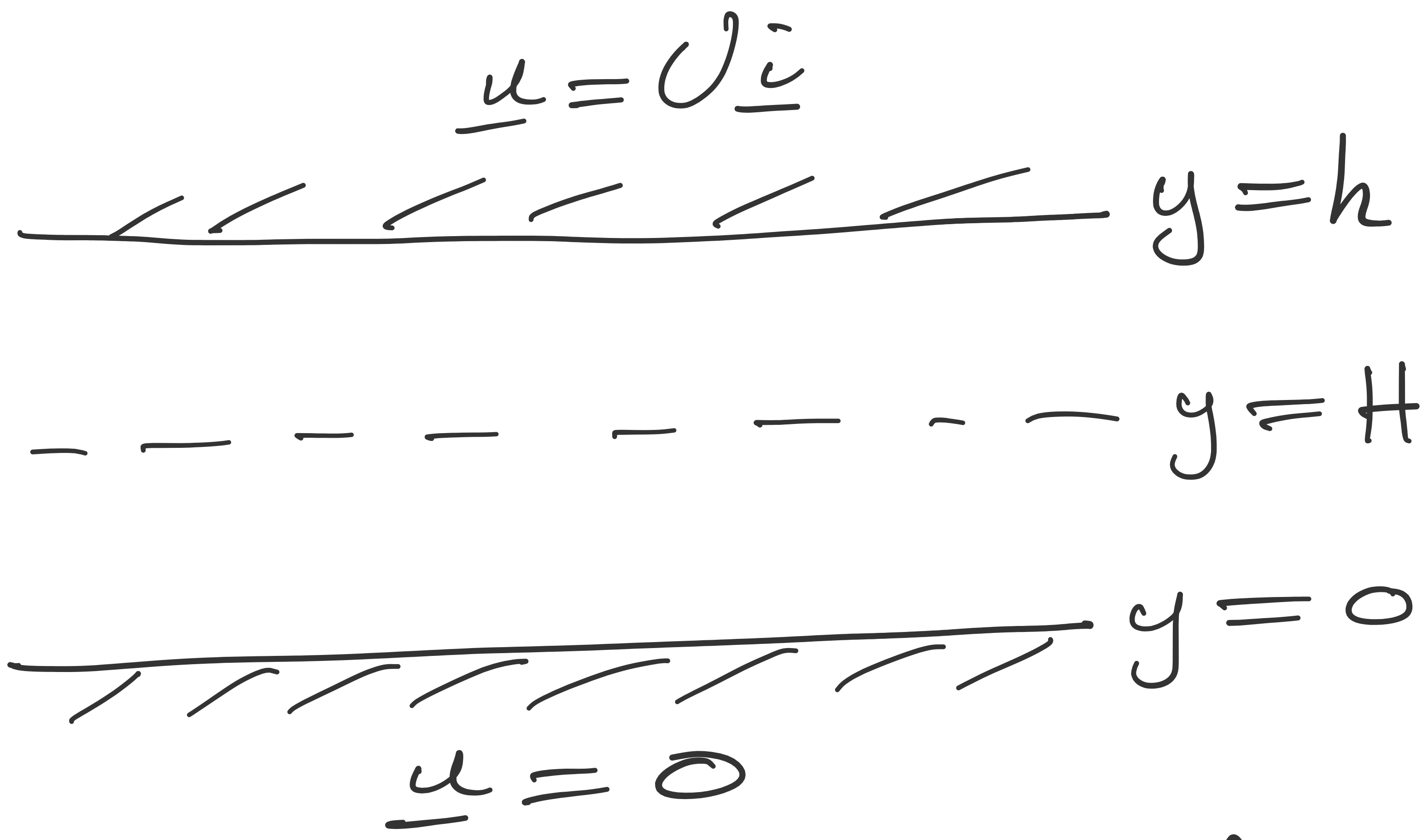
Couette flow with $u = u(y)$, $G = 0$



For steady flow, the 2D diffusion equation becomes just $\frac{d^2u}{dy^2} = 0$.

No-flux BCs on $y = 0, h$ are satisfied automatically.

No-slip BCs $\Rightarrow u(0) = 0$, $u(h) = U$

$\therefore \quad u(y) = U \frac{y}{h}$, a linear profile

The fluid above $y = H$ exerts a shear stress

$$\sigma_{12} = \mu \left.\frac{du}{dy}\right|_{y=H} = \mu \frac{U}{h}$$

on the fluid below $y = H$ (and vice versa).

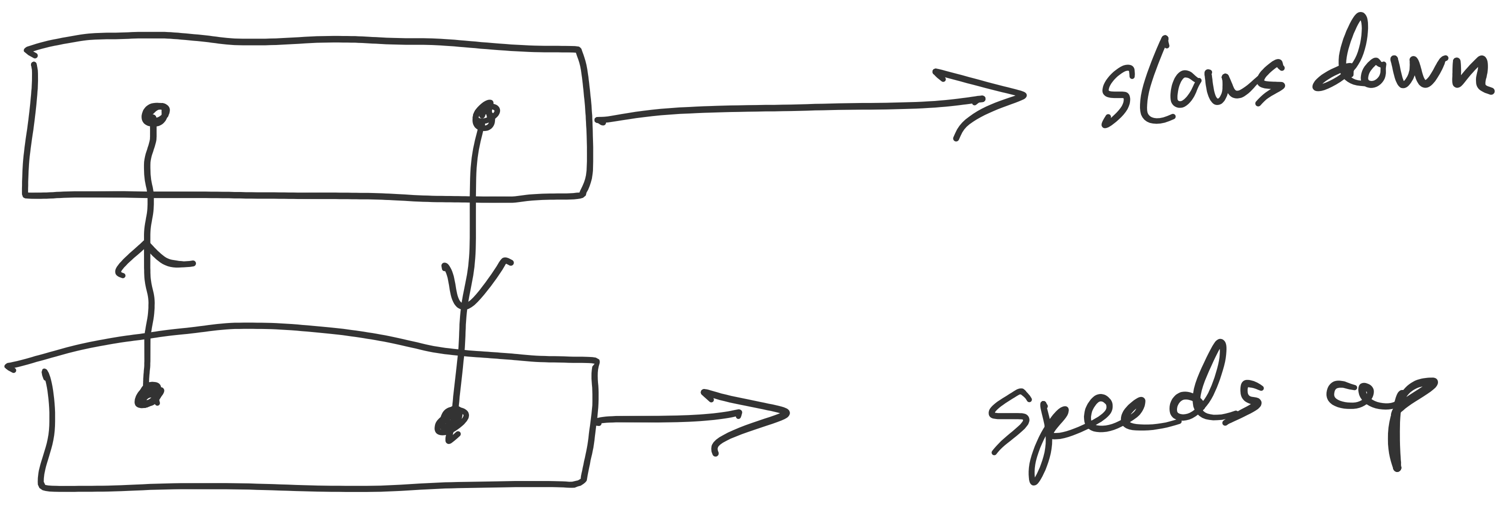
The shear stress is uniform because $\frac{Du}{Dt} = 0$, and we can "integrate"

$$0 = \nabla \cdot \underline{\underline{\sigma}} = \underline{e}_i \frac{\partial \sigma_{ij}}{\partial x_j}$$

to find that σ_{12} is uniform (constant).

Viscosity causes faster moving fluid above $y = H$ to "drag along" slower moving fluid below $y = H$.

By contrast $u(y)$ could be arbitrary in an inviscid fluid.



Think of people jumping between two points (or molecules moving in the y direction in a fluid) which slows down the faster moving fluid and speeds up the slower moving fluid.

Dimensionless Navier–Stokes equations

Transforming a problem into dimensionless variables is very illuminating for all areas of mathematical modelling.

For example if the fluid velocity scale U and sound speed c_s are such that the Mach number $Ma = U/c_s \ll 1$, we can safely ignore compressibility.

Consider an incompressible flow with far-field velocity $U\underline{i}$ around a stationary obstacle D with boundary ∂D of typical lengthscale (size) L.

$\underline{u} \to U\underline{i}$ as $|\underline{x}| \to \infty$

$\underline{u} = 0$ on ∂D

The Navier–Stokes equations are

(NS1) $\quad \nabla \cdot \underline{u} = 0$

(NS2) $\quad \rho \dfrac{D\underline{u}}{Dt} = -\nabla p + \mu \nabla^2 \underline{u}$

Nondimensionalise by scaling

$$\underline{x} = L\hat{\underline{x}}, \quad \underline{u} = U\hat{\underline{u}}, \quad t = \frac{L}{U}\hat{t}$$

with $\hat{\underline{x}}, \hat{\underline{u}}, \hat{t}$ dimensionless.

$$[\underline{x}] = L, \quad [\underline{u}] = U, \quad \left[\frac{\partial}{\partial t}\right] = [\underline{u}\cdot\nabla]$$

advective timescale

$\dfrac{D}{Dt}$ scales as one object

$$p = p_{atm} + [p]\hat{p}$$

atmospheric — pressure scale to be determined

$$x_i = L\hat{x}_i \Rightarrow \nabla = e_i \frac{\partial}{\partial x_i} = \frac{1}{L} e_i \frac{\partial}{\partial \hat{x}_i} = \frac{1}{L}\hat{\nabla}$$

(NS1) $\quad \dfrac{1}{L}\hat{\nabla}\cdot(U\hat{\underline{u}}) = 0 \Rightarrow \hat{\nabla}\cdot\hat{\underline{u}} = 0 \quad$ (NS1*)

(NS2) $$\frac{\rho U}{L/U}\frac{\partial \hat{\underline{u}}}{\partial \hat{t}} + \frac{\rho U^2}{L}\hat{\underline{u}}\cdot\hat{\nabla}\hat{\underline{u}} = -\frac{[p]}{L}\hat{\nabla}\hat{p} + \frac{\mu U}{L^2}\hat{\nabla}^2\hat{\underline{u}}$$

The advective scaling for time gives the same prefactor for $\dfrac{\partial \hat{\underline{u}}}{\partial t}$ and $\hat{\underline{u}}\cdot\hat{\nabla}\hat{\underline{u}}$

$$\frac{[\text{inertial term}]}{[\text{viscous term}]} = \frac{\rho U^2/L}{\mu U/L^2} = \frac{\rho U L}{\mu} = \frac{LU}{\nu} = Re$$

This dimensionless parameter is called the Reynolds number.

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Two natural regimes to explore using asymptotic methods for $Re \gg 1$ and $Re \ll 1$.

i) $Re \gg 1$

Choose an inviscid pressure scale $[p] = \rho U^2$

$$\Rightarrow \hat{\nabla} \cdot \hat{\underline{u}} = 0, \quad \frac{\partial \hat{\underline{u}}}{\partial \hat{t}} + \hat{\underline{u}} \cdot \hat{\nabla} \hat{\underline{u}} = -\hat{\nabla} \hat{p} + \underbrace{\frac{1}{Re} \hat{\nabla}^2 \hat{\underline{u}}}_{\text{small}}$$

Hope to ignore small viscous terms and solve the Euler equations, outside thin "boundary layers" where we need to keep the viscous term to satisfy no-slip BC.

ii) $Re \ll 1$

Choose a viscous pressure scale $[p] = \frac{\mu U}{L}$ to get

$$\hat{\nabla} \cdot \hat{\underline{u}} = 0, \quad \underbrace{Re \left(\frac{\partial \hat{\underline{u}}}{\partial \hat{t}} + \hat{\underline{u}} \cdot \hat{\nabla} \hat{\underline{u}} \right)}_{\text{small}} = -\hat{\nabla} \hat{p} + \hat{\nabla}^2 \hat{\underline{u}}$$

Hope to ignore small inertial terms and the solve the slow viscous flow equations (linear)

$$\hat{\nabla} \cdot \hat{\underline{u}} = 0, \quad \hat{\nabla}^2 \hat{\underline{u}} = \hat{\nabla} \hat{p}.$$

We will sometimes need to restore inertia in the "far field" at large lengthscales.

Two flows are <u>dynamically similar</u> if they satisfy the same dimensionless problem – used to scale real world flows into the lab.