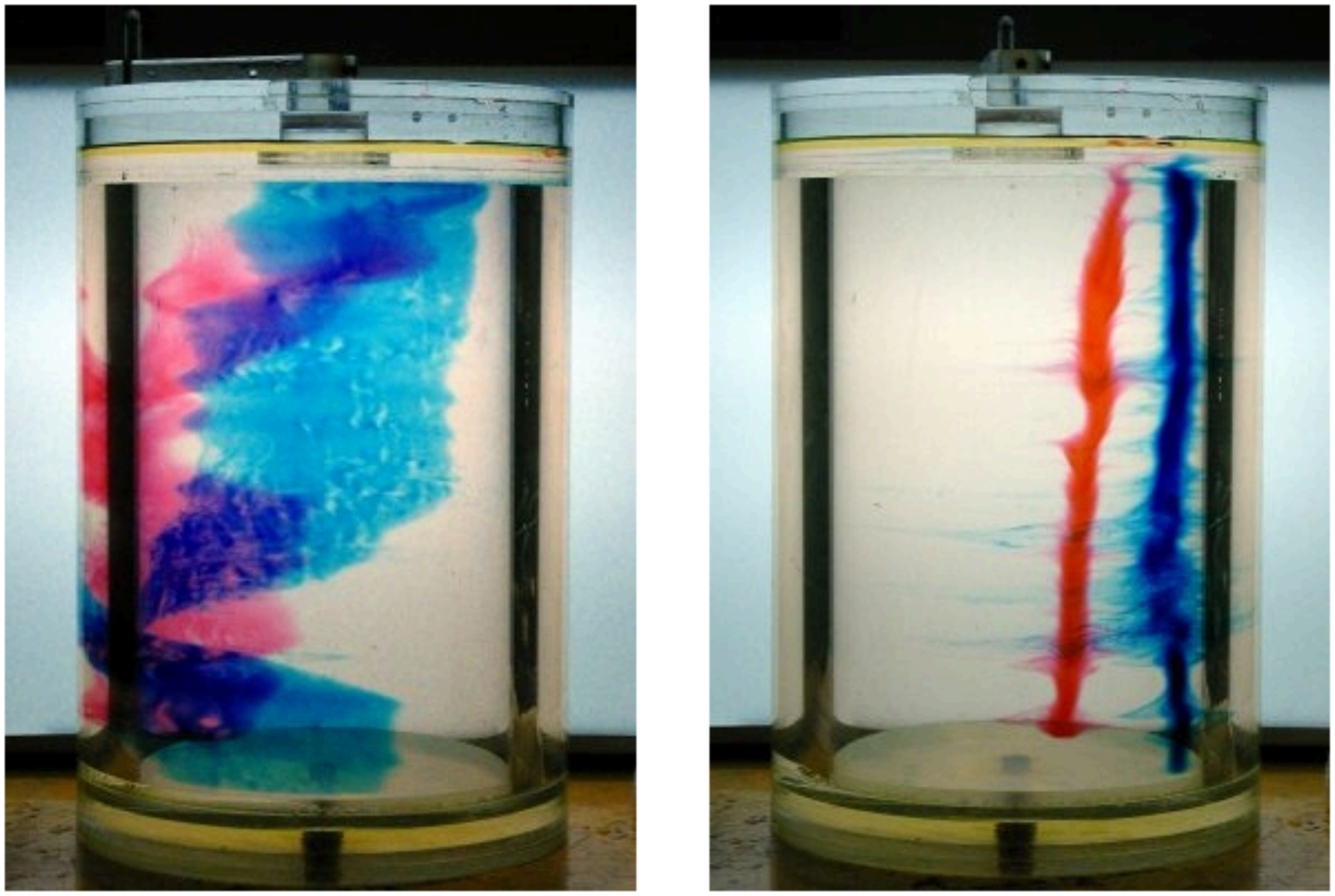
Monday, 16 November 2020 14:30

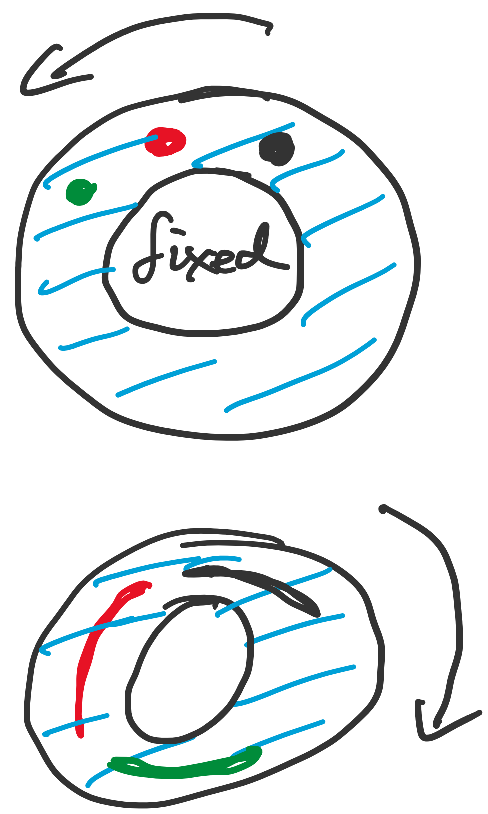
Viscous Flow lecture 11

Chapter 3: Low Reynolds number flows

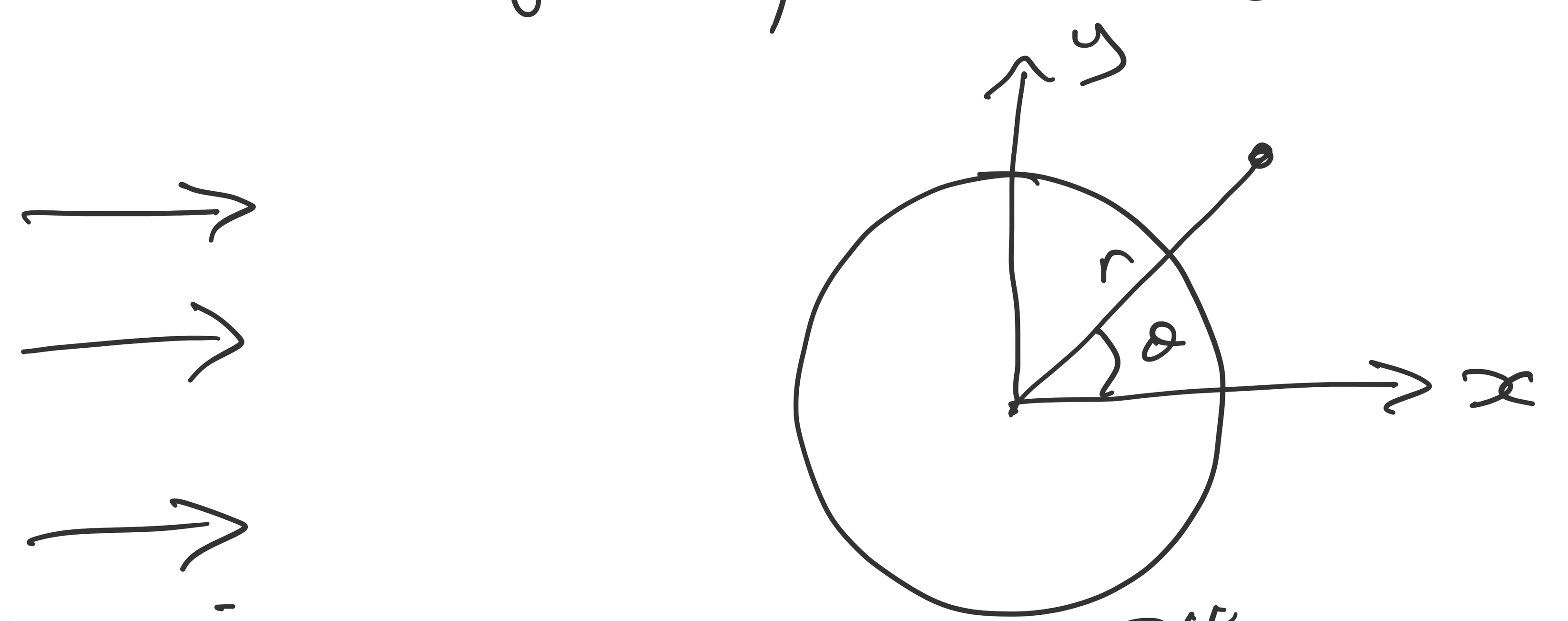


https://sciencedemonstrations.fas.harvard.edu/presentations/reversible-fluid-mixing

Flows at zero Reynolds number are time-reversible:



Slow viscous flow past a cylinder



$\underline{u} \sim \underline{i}$

$\psi \sim y = r\sin\theta$ as $r \to \infty$

$\psi = \frac{\partial \psi}{\partial r} = 0$ on $r=1$

Dimensionless problem for streamfunction ψ

Choose the Reynolds number based on the cylinder's radius a (so $r_{dim} = ar$).

We want the 2D NSE with

$$\underline{u} = \nabla \wedge (\psi \underline{k}) = \nabla\psi \wedge \underline{k}$$
$$= \psi_y \underline{i} - \psi_x \underline{j} \text{ in Cartesians}$$
$$= \frac{1}{r}\psi_\theta \underline{e_r} - \psi_r \underline{e_\theta} \text{ in polars}$$

using $\nabla\psi = \psi_r \underline{e_r} + \frac{1}{r}\psi_\theta \underline{e_\theta}$.

$$Re \frac{\partial(\psi, \nabla^2\psi)}{\partial(y, x)} = \nabla^4 \psi$$

note y first, then x

$$Re \frac{1}{r} \frac{\partial(\psi, \nabla^2\psi)}{\partial(\theta, r)} = \nabla^4\psi$$

where $\nabla^2 = \partial_{xx} + \partial_{yy}$
$$= \partial_r^2 + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_\theta^2$$

Seek an asymptotic solution for ψ at $r = O(1)$ as $Re \to 0$.

Try $\psi \sim \psi_0 + Re\,\psi_1 + \dots$ as $Re \to 0$

At leading order we get the biharmonic equation $\nabla^4 \psi_0 = 0$ (SF1)

BC on the cylinder $\psi_0 = \frac{\partial \psi_0}{\partial r} = 0$ on $r=1$ (SF2)

Far field: $\psi_0 \sim r\sin\theta$ as $r \to \infty$ (SF3)

This far-field behaviour suggests trying a separable solution
$$\psi_0(r,\theta) = f(r)\sin\theta.$$

$$\nabla^2\psi_0 = \left(\partial_r^2 + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_\theta^2\right) f(r)\sin\theta$$
$$= \left(\partial_r^2 + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\right) f(r)\sin\theta$$

$$\therefore\ 0 = \nabla^4\psi_0 = \nabla^2(\nabla^2\psi_0)$$
$$= \left(\partial_r^2 + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\right)^2 f(r)\sin\theta$$

This gives an ODE for f:
$$\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{1}{r^2}\right) f(r) = 0$$

This ODE is homogeneous in r & $\frac{d}{dr}$ so it's invariant under a scaling $r \mapsto \alpha r$ for constant α.

Either change variable to $s = \log r$ or try r^n as a solution.

This gives $n = -1, 1, 1, 3$ so
$$f(r) = \frac{A}{r} + Br + Cr\log r + Dr^3$$

Apply BCs:

$$0 = f(1) = A + B + D$$
$$0 = f'(1) = -A + B + C + 3D$$
$$1 = f'(\infty) \Rightarrow B = 1,\quad C = D = 0$$

$C = D = 0 \Rightarrow A = B = 0$ from the BC at $r=1$, so there's no solution.

There is no separable solution to $\nabla^4\psi_0 = 0$ that satisfies (SF2,3).

One can prove rigorously that there is no solution of any kind (not necessarily separable) to (SF1-3).

This is the Stokes paradox (1851). Resolved in 1957 by Proudman & Pearson.

Resolution of the Stokes paradox

The slow flow approximation is only valid for $r \ll 1/Re$.

Inertia returns in a "boundary layer at infinity"!

Rescale $x = \frac{1}{Re}\hat{x}$, $y = \frac{1}{Re}\hat{y}$

$r = \frac{1}{Re}\hat{r}$, $\psi = \frac{1}{Re}\hat{\psi}$

The scaling for ψ preserves

$\underline{u} = \nabla\psi \wedge \underline{k} = \hat{\nabla}\hat{\psi} \wedge \underline{k}$

This gives

$$\frac{\partial(\hat{\psi}, \hat{\nabla}^2\hat{\psi})}{\partial(\hat{y}, \hat{x})} = \hat{\nabla}^4\hat{\psi}$$

$\hat{y}$ first

where $\hat{\nabla}^2 = \partial_{\hat{x}}^2 + \partial_{\hat{y}}^2$.

This is the full 2D NSE with no large or small parameter left.

But – far from the cylinder the flow should look like a uniform stream:

$$\hat{\psi} \sim \hat{y} + \delta(Re)\hat{\psi}_1 + \ldots$$

where $\delta(Re) \to 0$ as $Re \to 0$

and $\hat{\psi}_1 = O(1)$.

(1910)

At $O(\delta)$ we get Oseen's equation:

$$\partial_{\hat{x}}\hat{\nabla}^2\hat{\psi}_1 = \hat{\nabla}^4\hat{\psi}_1.$$

This is equivalent to linearising the $\underline{u}\cdot\nabla\underline{u}$ term into $\underline{u}_0\cdot\nabla\underline{u}_1$ for $\underline{u}_0 = \underline{i}$, $\underline{u} = \underline{u}_0 + \delta(Re)\,\underline{u}_1 + \ldots$

Oseen's equation, with a point force at the origin representing the cylinder, is linear with constant coefficients so in principle it can be solved using a Fourier transform to find a solution satisfying $\hat{\psi}_1 \to 0$ as $\hat{r} \to \infty$, and

$$\hat{\psi}_1 \sim E\,\hat{r}\log\hat{r}\,\sin\theta \text{ as } \hat{r} \to 0.$$

Matching:

There are logs, so we can match by "intermediate variable". We require the two expansions to agree in an intermediate region where $r = Re^{-\alpha}\tilde{r}$ with $\tilde{r} = O(1)$ and $0 < \alpha < 1$.

$\alpha = 0$ is the "inner" problem near the cylinder

$\alpha = 1$ is the boundary layer at infinity

Details are online in and not examinable. Also in Hinch "Perturbation Methods" §5.2 and §5.3.

The matching gives

$$\delta(Re) = \frac{1}{\log(1/Re)}$$

and (SF3) becomes the matching condition

$$\psi_0 \sim \delta(Re)\, r\log r \sin\theta \quad \text{as } r \to \infty.$$

Plausibility argument for this scaling.

General solution for $r = O(1)$ is $\psi_0 = f(r)\sin\theta$ with

$$f(r) = \frac{A}{r} + Br + Cr\log r + Dr^3.$$

The solution that is least badly behaved as $r \to \infty$, and satisfies $f(1) = f'(1) = 0$, is

$$\psi_0(r,\theta) = C\left(r\log r - \frac{r}{2} + \frac{1}{2r}\right)\sin\theta.$$

Putting $r = \hat{r}/\varepsilon$ with $\varepsilon = Re$ and $\hat{r} = O(1)$ gives

$$\psi_0 \sim C\,(\hat{r}/\varepsilon)\log(\hat{r}/\varepsilon)\sin\theta$$

$$\sim C\,(\hat{r}/\varepsilon)\left[\log\hat{r} + \log\tfrac{1}{\varepsilon}\right]\sin\theta$$

$$\sim C\log\tfrac{1}{\varepsilon}\,\underbrace{\left(\frac{\hat{r}}{\varepsilon}\right)\sin\theta}_{= r\sin\theta = y}$$

as $\varepsilon \to 0$ with $\hat{r}$ fixed.

In this limit we keep $\log 1/\varepsilon$ and discard $\log\hat{r}$ from $[\log\hat{r} + \log\frac{1}{\varepsilon}]$

This gives $\psi_0 \sim y = r\sin\theta$ as $\varepsilon \to 0$ for fixed $\hat{r} = \varepsilon r$ provided $C\log\frac{1}{\varepsilon} = 1$, i.e.

$C = \dfrac{1}{\log 1/\varepsilon}$, as one would get by matching using intermediate variables.

Hence

$$\psi_0(r,\theta) = \frac{1}{\log\left(\frac{1}{Re}\right)}\left(r\log r - \frac{r}{2} + \frac{1}{2r}\right)\sin\theta$$

for $r = O(1)$.

The next term ψ_1 would be $O\left(\log(1/Re)^{-2}\right)$ which gives very slow convergence.

The dimensionless drag on the cylinder (exercise) $\sim \dfrac{4\pi}{\log(1/Re)}$ per unit length.