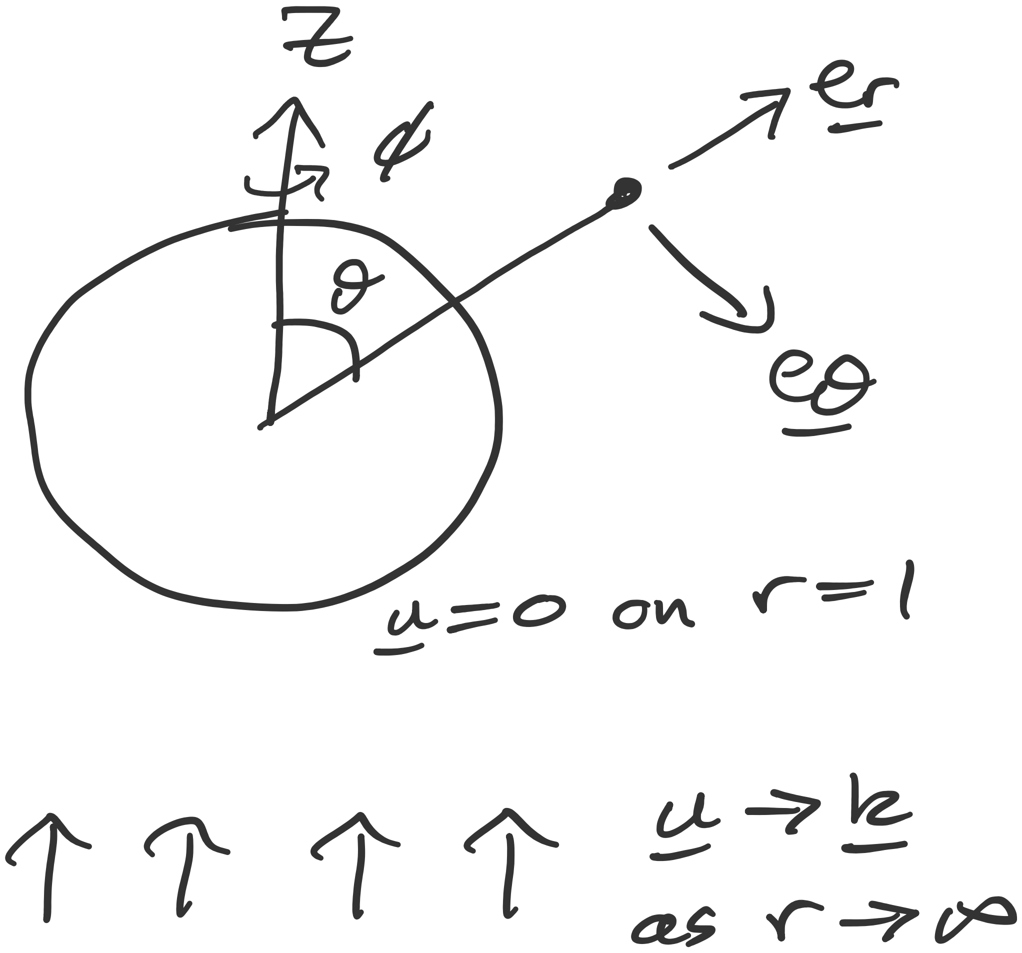
Tuesday, 17 November 2020 18:41

# Viscous Flow Lecture 12

## Slow flow past a sphere

Axisymmetric flow
$\underline{u}(r,\theta) = u_r \underline{e_r} + u_\theta \underline{e_\theta}$

and $p(r,\theta)$ in spherical polar coordinates.




Dimensionless variables

$$\underline{u} \to \underline{k} = \underline{e_r}\cos\theta - \underline{e_\theta}\sin\theta \quad \text{as } r \to \infty$$

$$\nabla \cdot \underline{u} = 0 \quad \text{and} \quad Re\,(\underline{u}\cdot\nabla)\underline{u} = -\nabla p + \nabla^2 \underline{u}$$

Expand
$$\underline{u} = \underline{u}^{(0)} + Re\,\underline{u}^{(1)} + \dots$$
$$p = p^{(0)} + Re\,p^{(1)} + \dots$$

Keep only the leading order terms $\underline{u}^{(0)}$ and $p^{(0)}$ and drop "(0)" superscripts.

Slow (Stokes) flow equations

$$\nabla\cdot\underline{u} = 0 \quad \text{and} \quad \nabla p = \nabla^2 \underline{u}$$

Tuesday, 17 November 2020 18:45

In spherical polar coordinates, incompressibility becomes

$$0 = \nabla \cdot \underline{u}$$
$$= \frac{1}{r^2 \sin\theta}\left[\frac{\partial}{\partial r}\left(u_r r^2 \sin\theta\right) + \frac{1}{r}\frac{\partial}{\partial \theta}\left(u_\theta r^2 \sin\theta\right)\right]$$

Leaving r

often cancelled

where $r^2 \sin\theta$ is the Jacobian for $(x, y, z) \mapsto (r, \theta, \phi)$ coordinates

Written like this as $\frac{\partial}{\partial r}$ and $\frac{1}{r}\frac{\partial}{\partial \theta}$ come from $\nabla$ in spherical polars.

We can satisfy incompressibility by introducing a Stokes streamfunction $\psi(r, \theta)$ s.t.

$$u_r = \frac{1}{r^2 \sin\theta}\frac{\partial \psi}{\partial \theta}, \quad u_\theta = -\frac{1}{r\sin\theta}\frac{\partial \psi}{\partial r}$$

In vectors,

$$\underline{u} = \frac{1}{r^2 \sin\theta}\begin{vmatrix} \underline{e_r} & r\underline{e_\theta} & r\sin\theta\, \underline{e_\phi} \\ \partial_r & \partial_\theta & \partial_\phi \\ 0 & 0 & \psi(r,\theta) \end{vmatrix}$$

$$= \nabla \wedge \left(\frac{\psi}{r\sin\theta}\underline{e_\phi}\right)$$

This is the spherical analogue of $\underline{u} = \nabla \wedge (\psi \underline{k})$ in 2D.

$r\sin\theta\, d\phi$ is a length on a latitude circle from varying $\phi$


To eliminate the pressure we'll want the vorticity equation.

$$\underline{\omega} = \nabla \wedge \underline{u} = \frac{1}{r^2 \sin\theta}\begin{vmatrix} \underline{e_r} & r\underline{e_\theta} & r\sin\theta\, \underline{e_\phi} \\ \partial_r & \partial_\theta & \partial_\phi \\ u_r & r u_\theta & r\sin\theta\, u_\phi \end{vmatrix}$$

$$= -\underline{e_\phi}\frac{1}{r\sin\theta}\left(\frac{\partial^2 \psi}{\partial r^2} - \frac{\cot\theta}{r^2}\frac{\partial \psi}{\partial \theta} + \frac{1}{r^2}\frac{\partial^2 \psi}{\partial \theta^2}\right)$$

$$= -\underline{e_\phi}\frac{1}{r\sin\theta}D^2\psi$$

This defines the Stokes operator $D^2$.
Note that $D^2 \neq \nabla^2$ because $D^2$ is applied to vector fields in the $\underline{e_\phi}$ direction, and $\underline{e_\phi}$ is itself a function of $\theta$ and $\phi$.

The slow flow equations are

$$\nabla p = \nabla^2 \underline{u} = \nabla(\nabla \cdot \underline{u}) - \nabla \wedge (\nabla \wedge \underline{u})$$
$$= -\mathrm{curl}^2 \underline{u} \quad \text{as } \nabla \cdot \underline{u} = 0$$

Taking the curl again to eliminate the pressure gives

$$0 = \mathrm{curl}(\nabla p) = -\mathrm{curl}^3 \underline{u}$$
$$= -\mathrm{curl}^4\left(\frac{\psi}{r\sin\theta}\underline{e_\phi}\right)$$

$$\Rightarrow D^4 \psi = 0 \qquad (D^4 = (D^2)^2)$$

In 2D Cartesians we had $\nabla^4 \psi = 0$

Boundary conditions:

$$\underline{u} = 0 \text{ on } r = 1 \Rightarrow \psi = \frac{\partial \psi}{\partial r} = 0$$

The far field condition for uniform oncoming flow is

$$\underline{u} \to \underline{k} = \underline{e_r}\cos\theta - \underline{e_\theta}\sin\theta$$

as $r \to \infty$, so

$$u_r = \frac{1}{r^2 \sin\theta}\frac{\partial \psi}{\partial \theta} \to \cos\theta$$
$$u_\theta = -\frac{1}{r\sin\theta}\frac{\partial \psi}{\partial r} \to -\sin\theta,$$

giving $\psi \sim \frac{1}{2} r^2 \sin^2\theta$ as $r \to \infty$

This behaviour in the far field suggests the separable solution

$$\psi(r, \theta) = f(r)\sin^2\theta$$

$$D^2 \psi = \left(\frac{\partial^2}{\partial r^2} - \frac{2}{r^2}\right) f(r) \sin^2\theta$$

$$D^4 \psi = \left(\frac{\partial^2}{\partial r^2} - \frac{2}{r^2}\right)^2 f(r) \sin^2\theta$$

This is homogeneous in $r$, so try $f(r) = r^n$ for solutions.

This gives $r^4, r^2, r, 1/r$

Applying the boundary conditions

$f = f' = 0$ on $r = 1$,

$f(r) \sim \frac{1}{2} r^2$ as $r \to \infty$,

gives $$\psi(r, \theta) = \left(\frac{1}{2} r^2 - \frac{3}{4} r + \frac{1}{4r}\right)\sin^2\theta.$$

The straightforward solution works at leading order for flow around a sphere.

A paradox arises if we try to find the $O(Re)$ corrections $\underline{u}^{(1)}$ and $p^{(1)}$ since $\psi_1$ has an $r^2 \sin^2\theta \cos\theta$ dependence that can't match to the uniform flow at infinity.
Again, we need a rescaling for a boundary layer at infinity.

This version is called the Whitehead paradox.

Tuesday, 17 November 2020  18:45

# The Stokes drag formula (1851)

Involved in at least 3 Nobel Prize – winning research projects.

$$\psi = \left(\tfrac{1}{2}r^2 - \tfrac{3}{4}r + \tfrac{1}{4r}\right)\sin^2\theta$$

The velocity components are:

$$u_r = \frac{1}{r^2\sin\theta}\frac{\partial\psi}{\partial\theta} = \left(1 - \frac{3}{2r} + \frac{1}{2r^3}\right)\cos\theta$$

$$u_\theta = -\frac{1}{r\sin\theta}\frac{\partial\psi}{\partial r} = \left(-1 + \frac{3}{4r} + \frac{1}{4r^3}\right)\sin\theta$$

From $\nabla p = -\operatorname{curl}^2 \underline{u}$ we get

$$p = p_\infty - \frac{3\cos\theta}{2r^2}$$

with $p_\infty$ the arbitrary constant pressure at infinity.

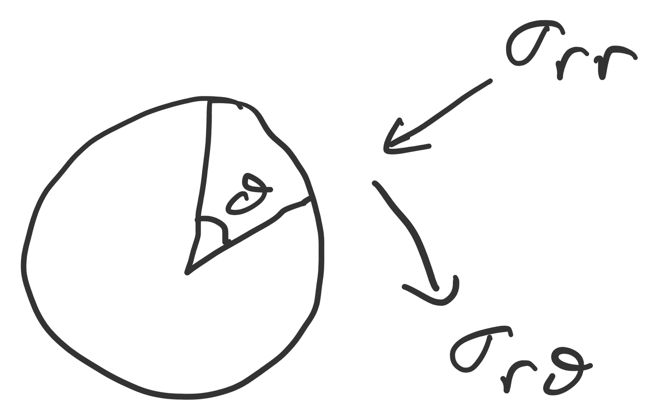
$$p - p_\infty = O(1/r^2),$$

$$|\underline{u} - \underline{k}| = O(1/r) \quad \text{with } \underline{u}_\infty = \underline{k}$$

The drag force is in the $\underline{k}$ direction by symmetry, so equal to $D\,\underline{k}$ where

$$D = \iint_{r=1} \underline{k}\cdot\underline{\underline{\sigma}}\cdot\underline{n}\, dS$$

$\underline{n}$ outward normal




$$\underline{k} = \underline{e_r}\cos\theta - \underline{e_\theta}\sin\theta$$

$$D = \iint_{r=1} \sigma_{rr}\cos\theta - \sigma_{r\theta}\sin\theta\, dS$$

where

$$\sigma_{rr} = -p + \frac{2\partial u_r}{\partial r}$$

$$\sigma_{rr}\Big|_{r=1} = -p_\infty + \tfrac{3}{2}\cos\theta$$

$$\sigma_{r\theta} = r\frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right) + \frac{1}{r}\frac{\partial u_r}{\partial\theta}$$

$$\sigma_{r\theta}\Big|_{r=1} = -\tfrac{3}{2}\sin\theta$$

$$\therefore\ D = \int_0^{2\pi} d\phi \int_0^{\pi} d\theta\, \sin\theta\left(\sigma_{rr}\cos\theta - \sigma_{r\theta}\sin\theta\right)$$

$$= -p_\infty\cos\theta + 3/2$$

The viscous contribution is constant, and the pressure contribution integrates to zero.

$$D = 4\pi \times \frac{3}{2} = 6\pi$$

(Area is $4\pi$)

Restoring the dimensions gives Stokes' famous drag force

$$D = 6\pi\underbrace{\left(\frac{\mu U}{a}\right)}_{\text{scale for } \underline{\underline{\sigma}} \text{ and } p}\underbrace{a^2}_{\text{scale for area}} = 6\pi\mu a U$$

For example, the terminal velocity of a solid sphere of uniform density $\rho_s$ falling through viscous fluid with density $\rho_L$ is given by

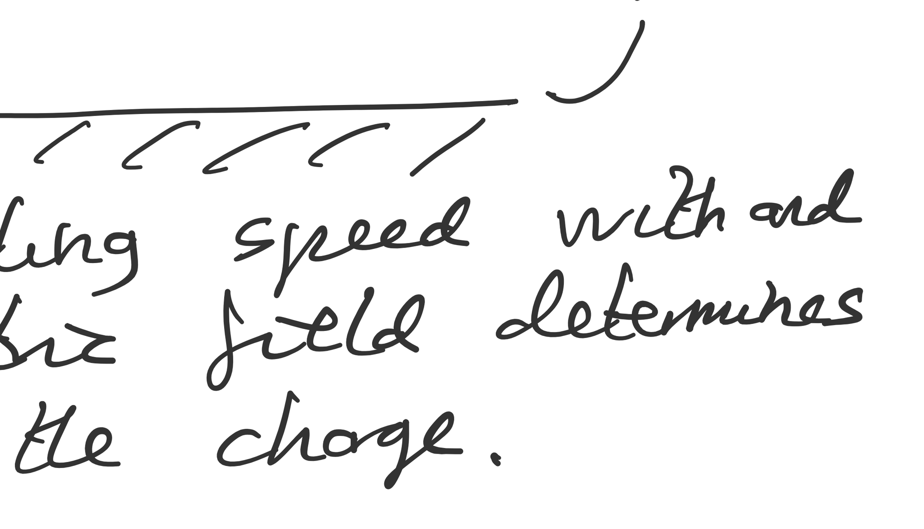
$$6\pi\mu a U = \tfrac{4}{3}\pi a^3(\rho_s - \rho_L)g$$

so

$$U = \frac{2}{9}\,\frac{a^2(\rho_s - \rho_L)g}{\mu}.$$

This can be used to measure viscosities.

This can also be used to measure the electric charge on a drop of oil in air




Measuring the falling speed with and without the electric field determines the radius and the charge.

Millikan & Fletcher measured the charge on the electron like this to get (1913)

$$e = 1.5924(17)\times 10^{-19}\,C$$

The modern figure is

$$e = 1.6022\ldots \times 10^{-19}\,C.$$

The Stokes drag formula was also used in Einstein's theory of Brownian motion. Verified experimentally by Perrin.