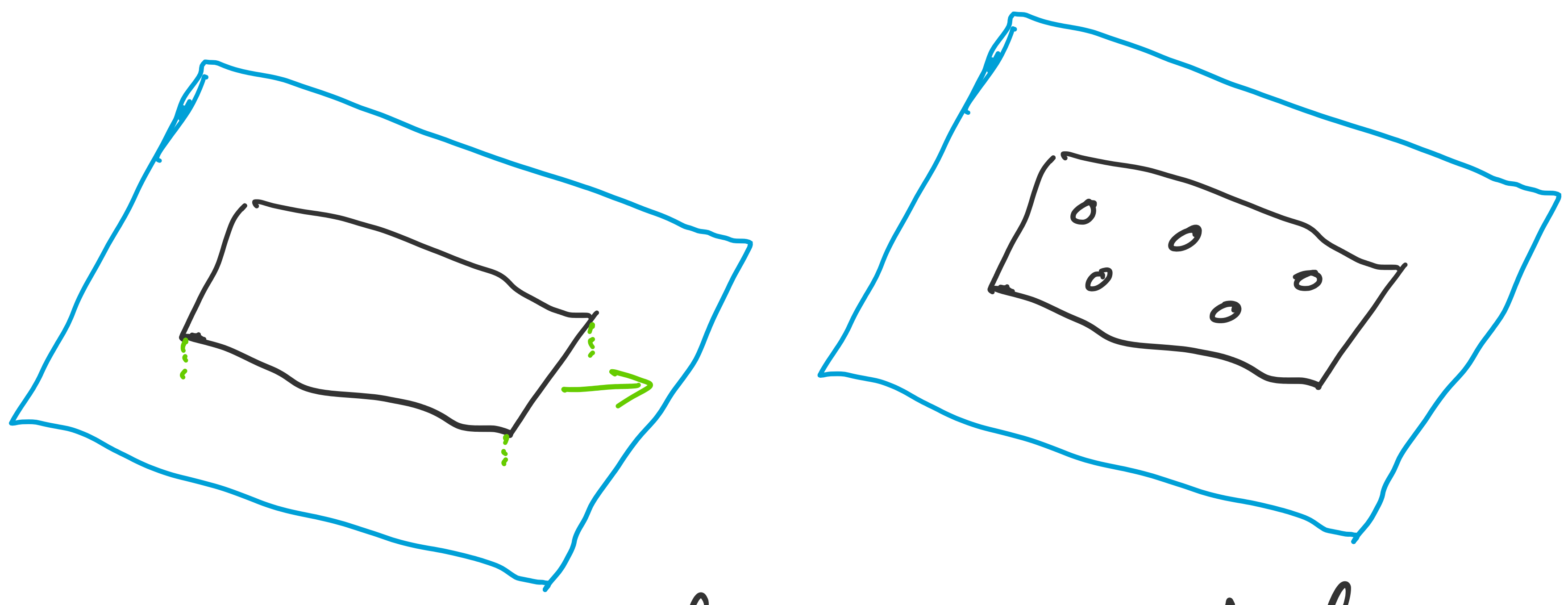
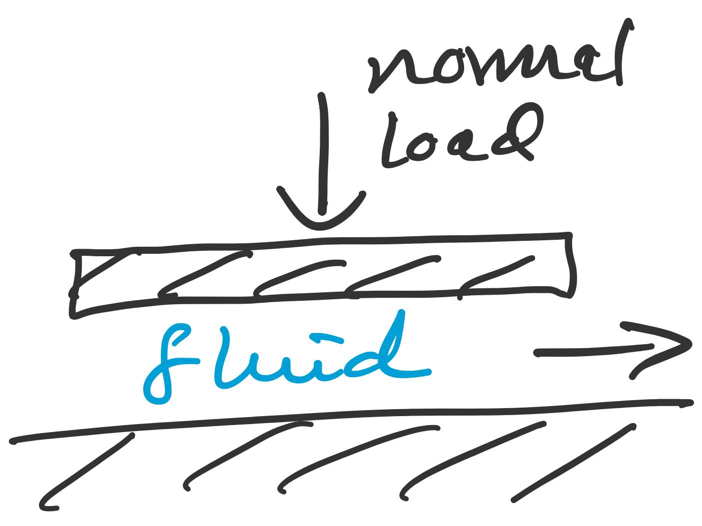
Viscous Flow Lecture 13
Lubrication theory



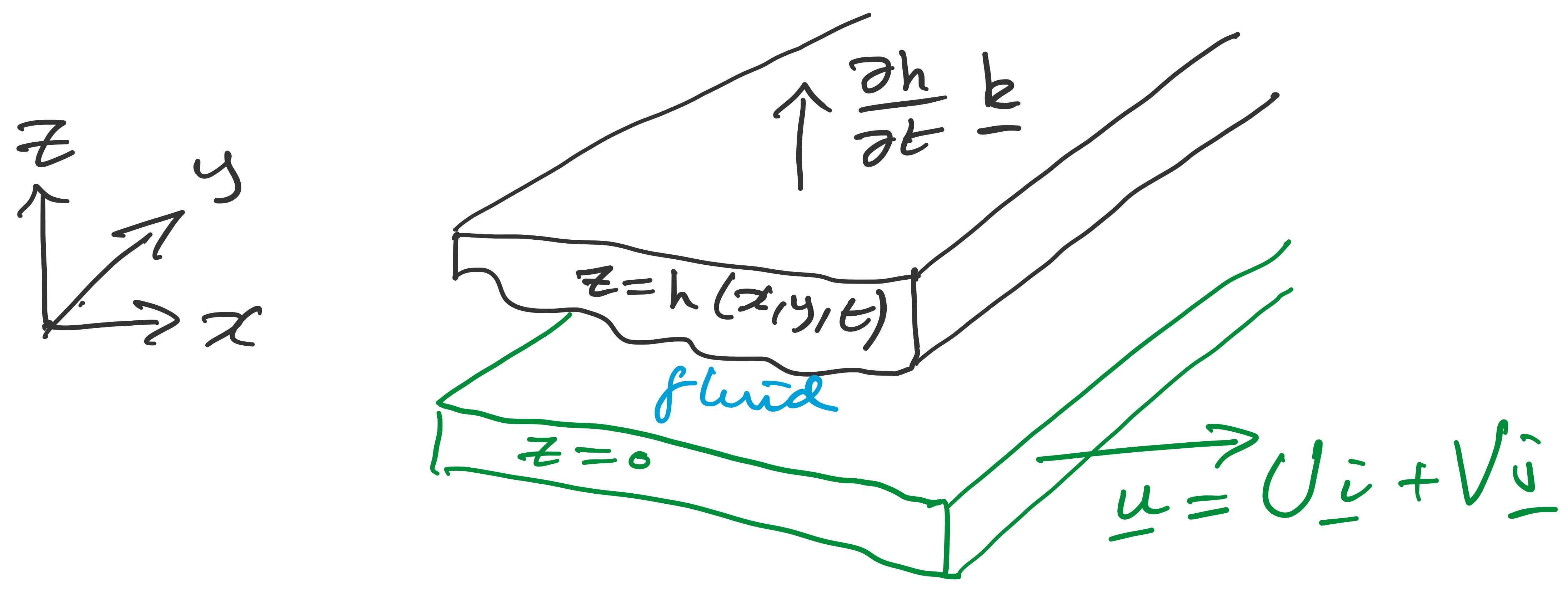
A sheet of paper glides across a flat table... unless you punch holes in the paper.

Lubrication theory explains how a thin film (or layer) of viscous fluid can support a large normal load with little resistance to transverse motion.

It combines ideas from boundary layers with the slow viscous flow equations



Canonical problem



Consider flow in a thin layer $0 < z < h(x,y,t)$ with prescribed layer thickness $h(x,y,t)$. Suppose the plate at $z=0$ moves horizontally with velocity $U\underline{i} + V\underline{j}$, and the upper plate moves vertically with velocity $\frac{\partial h}{\partial t}\underline{k}$.

Incompressible Navier-Stokes:

$$\rho \frac{D\underline{u}}{Dt} = -\nabla p + \mu \nabla^2 \underline{u} \qquad (1)$$

$$\nabla \cdot \underline{u} = 0 \qquad (2)$$

No-flux and no-slip boundary conditions:

$$u = U, \quad v = V, \quad w = 0 \quad \text{on } z = 0 \qquad (3)$$

$$u = 0, \quad v = 0, \quad w = \frac{\partial h}{\partial t} \quad \text{on } z = h \qquad (4)$$

Sunday, 22 November 2020 21:33

Dimensionless equations

Suppose $[x]=[y]=L$
$[h]=\delta L$
$[u]=[v]=U_0$

with the aspect ratio $\delta \ll 1$.

Scale $x=L\hat{x},\ y=L\hat{y},\ z=\delta L\hat{z}$
$h=\delta L\hat{h}$

$u=U_0\hat{u},\ v=U_0\hat{v}$
$U=U_0\hat{U},\ V=U_0\hat{V}$ (lower boundary velocity)

(2) $\Rightarrow \left[\frac{\partial w}{\partial z}\right]=\left[\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right]$

$\Rightarrow w=\delta U_0\hat{w}$

(4z) $\Rightarrow \left[\frac{\partial h}{\partial t}\right]=[w] \Rightarrow t=\frac{L}{U_0}\hat{t}$

Finally, write the pressure as
$p=p_{atm}+[p]\hat{p}$ with pressure scale $[p]$ to be determined.

$$\frac{D}{Dt}=\frac{\partial}{\partial t}+\underline{u}\cdot\nabla=\frac{1}{L/U_0}\left(\frac{\partial}{\partial \hat{t}}+\hat{\underline{u}}\cdot\hat{\nabla}\right)$$
$$=\frac{U_0}{L}\frac{D}{D\hat{t}}$$

(2) $\times \frac{L}{U_0} \Rightarrow \frac{\partial \hat{u}}{\partial \hat{x}}+\frac{\partial \hat{v}}{\partial \hat{y}}+\frac{\partial \hat{w}}{\partial \hat{z}}=0$

(1x) $\times \frac{\delta^2 L}{\mu U_0} \Rightarrow \delta^2 Re\frac{D\hat{u}}{D\hat{t}}=-[p]\frac{\delta^2 L}{\mu U_0}\frac{\partial \hat{p}}{\partial \hat{x}}+\delta^2\left(\frac{\partial^2\hat{u}}{\partial\hat{x}^2}+\frac{\partial^2\hat{u}}{\partial\hat{y}^2}\right)+\frac{\partial^2\hat{u}}{\partial\hat{z}^2}$

(1y) $\times \frac{\delta^2 L}{\mu U_0} \Rightarrow \delta^2 Re\frac{D\hat{v}}{D\hat{t}}=-[p]\frac{\delta^2 L}{\mu U_0}\frac{\partial \hat{p}}{\partial \hat{y}}+\delta^2\left(\frac{\partial^2\hat{v}}{\partial\hat{x}^2}+\frac{\partial^2\hat{v}}{\partial\hat{y}^2}\right)+\frac{\partial^2\hat{v}}{\partial\hat{z}^2}$

where $Re=\frac{LU_0}{\nu}$

(1z) $\times \frac{\delta^3 L}{\mu U_0} \Rightarrow \delta^4 Re\frac{D\hat{w}}{D\hat{t}}=-[p]\frac{\delta^2 L}{\mu U_0}\frac{\partial \hat{p}}{\partial \hat{z}}+\delta^4\left(\frac{\partial^2\hat{w}}{\partial\hat{x}^2}+\frac{\partial^2\hat{w}}{\partial\hat{y}^2}\right)+\delta^2\frac{\partial^2\hat{w}}{\partial\hat{z}^2}$

Lubrication approximation

Lubrication theory assumes a small aspect ratio $\delta \ll 1$, and a small reduced Reynolds number $\delta^2 Re \ll 1$. The usual Reynolds number Re need not be small.

We get $\delta^2 Re$ because
$w\frac{\partial}{\partial z}\sim \delta U_0\frac{1}{\delta L}\sim\frac{U_0}{L}\sim u\frac{\partial}{\partial x}$

while $\frac{\partial^2}{\partial z^2}\sim\frac{1}{\delta^2 L^2}$

Choose $[p]=\frac{\mu U_0}{\delta^2 L}$ to balance the pressure gradient with the viscous terms in the x & y momentum equations.

At leading order (and dropping hats) we obtain the lubrication equations:

$$\frac{\partial^2 u}{\partial z^2}=\frac{\partial p}{\partial x},\quad \frac{\partial^2 v}{\partial z^2}=\frac{\partial p}{\partial y},\quad 0=\frac{\partial p}{\partial z} \qquad \text{(L1)}$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0. \qquad \text{(L2)}$$

The boundary conditions are unchanged at leading order:

$$u=U,\ v=V,\ w=0 \quad \text{on } z=0 \qquad \text{(L3)}$$
$$u=0,\ v=0,\ w=\frac{\partial h}{\partial t} \quad \text{on } z=h \qquad \text{(L4)}$$

Cross-layer averaged mass conservation

$$0 = \int_0^{h(x,y,t)} \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) dz$$

$$= \frac{\partial}{\partial x}\left(\int_0^h u\, dz \right) - u \frac{\partial h}{\partial x}\Big|_{z=h}$$

$$+ \frac{\partial}{\partial y}\left(\int_0^h v\, dz \right) - v \frac{\partial h}{\partial y}\Big|_{z=h}$$

$$+ [w]_0^h$$

using Leibniz's rule & the BC at $z=0$

$$0 = \frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(h\bar{u}) + \frac{\partial}{\partial y}(h\bar{v}) \quad *$$

where the depth-averaged horizontal velocity components are

$$\bar{u} = \frac{1}{h}\int_0^h u\, dz, \quad \bar{v} = \frac{1}{h}\int_0^h v\, dz$$

for $\bar{u}(x,y,t)$ and $u(x,y,z,t)$, similarly for $\bar{v}$ and v.

* Also holds when the fluid is bounded by free surfaces instead of rigid boundaries (Wu 1981).

Cross-layer-averaged momentum equation

L1z $\Rightarrow p = p(x,y,t)$ as $\frac{\partial p}{\partial z} = 0$.

L1x $\Rightarrow \frac{\partial^2 u}{\partial z^2} = \frac{\partial p}{\partial x}$ is independent of z

Integrating twice w.r.t. z and using the BC L3x and L4x gives

$$u = \underbrace{-\frac{1}{2}\frac{\partial p}{\partial x} z(h-z)}_{\text{Poiseuille flow}} + \underbrace{U(1-z/h)}_{\text{Couette flow}}$$

$$\bar{u} = \frac{1}{h}\int_0^h u\, dz = -\frac{h^2}{12}\frac{\partial p}{\partial x} + \frac{1}{2}U$$

Similarly

$$v = -\frac{1}{2}\frac{\partial p}{\partial y} z(h-z) + V(1-z/h)$$

$$\bar{v} = -\frac{h^2}{12}\frac{\partial p}{\partial y} + \frac{1}{2}V$$

Substituting these $\underline{u}$ and $\underline{v}$ into * gives Reynolds' lubrication equation

$$\left.\begin{aligned} &\frac{\partial}{\partial x}\left(\frac{h^3}{12}\frac{\partial p}{\partial x} \right) + \frac{\partial}{\partial y}\left(\frac{h^3}{12}\frac{\partial p}{\partial y} \right) \\ &= \frac{\partial h}{\partial t} + \frac{U}{2}\frac{\partial h}{\partial x} + \frac{V}{2}\frac{\partial h}{\partial y} \end{aligned}\right\} \text{(RLE)}$$

This is a self-adjoint elliptic equation for $p(x,y,t)$, parametric in t, given $h(x,y,t)$.

It's elliptic because (DEs 1)

$$\frac{h^3}{12}\frac{\partial^2 p}{\partial x^2} + 0\frac{\partial^2 p}{\partial x \partial y} + \frac{h^3}{12}\frac{\partial^2 p}{\partial y^2}$$

$$= -\frac{\partial}{\partial x}\left(\frac{h^3}{12}\right)\frac{\partial p}{\partial x} - \frac{\partial}{\partial y}\left(\frac{h^3}{12}\right)\frac{\partial p}{\partial y}$$

$$+ \frac{\partial h}{\partial t} + \frac{1}{2}U\frac{\partial h}{\partial x} + \frac{1}{2}V\frac{\partial h}{\partial y}$$

$$= F\left(x,y,t,p,\frac{\partial p}{\partial x},\frac{\partial p}{\partial y}\right)$$

The LHS is of the form

$$a(x,y)\frac{\partial^2 p}{\partial x^2} + 2b(x,y)\frac{\partial^2 p}{\partial x \partial y} + c(x,y)\frac{\partial^2 p}{\partial y^2}$$

with $a(x,y) = c(x,y) = \frac{h^3}{12}$

and $b(x,y) = 0$

so $ac > b^2 \Rightarrow$ elliptic

stress tensor

Apply the lubrication theory scalings to

$$\sigma_{ij} = -p\delta_{ij} + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) = \frac{\mu U_0}{\delta^2 L}\,\hat{\sigma}_{ij}.$$

This is the scaling that we used for pressure, to balance the viscous terms. Keep hats for clarity.

$$\hat{\sigma}_{11} = -p + 2\delta^2 \frac{\partial \hat{u}}{\partial \hat{x}}$$

$$\hat{\sigma}_{22} = -p + 2\delta^2 \frac{\partial \hat{v}}{\partial \hat{y}}$$

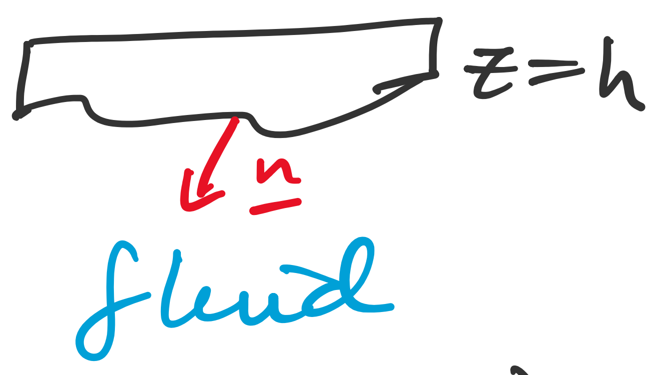
$$\hat{\sigma}_{33} = -p + 2\delta^2 \frac{\partial \hat{w}}{\partial \hat{z}}$$

$$\hat{\sigma}_{12} = \delta^2\left(\frac{\partial \hat{u}}{\partial \hat{y}} + \frac{\partial \hat{v}}{\partial \hat{x}}\right)$$

$$\hat{\sigma}_{13} = \delta \frac{\partial \hat{u}}{\partial \hat{z}} + \delta^3 \frac{\partial \hat{w}}{\partial \hat{x}}$$

$$\hat{\sigma}_{23} = \delta \frac{\partial \hat{v}}{\partial \hat{z}} + \delta^3 \frac{\partial \hat{w}}{\partial \hat{y}}$$

The stress on the upper boundary is $\underline{t}(\underline{n}) = e_i \sigma_{ij} n_j$ where $\underline{n}$ is the downwards (towards the fluid) pointing normal to the surface $z - h(x,y,t) = 0$.



$$\underline{n} = \frac{\nabla(h-z)}{|\nabla(h-z)|} = \frac{(h_x, h_y, -1)}{\sqrt{1+h_x^2+h_y^2}}$$

$$= \frac{(\delta \hat{h}_{\hat{x}}, \delta \hat{h}_{\hat{y}}, -1)}{\sqrt{1+\delta^2 \hat{h}_{\hat{x}}^2 + \delta^2 \hat{h}_{\hat{y}}^2}}$$

Scale $\underline{t} = \dfrac{\mu U_0}{\delta^2 L}\,\hat{\underline{t}}$

Find

$$\hat{\underline{t}} \sim -\delta\left(\hat{p}\hat{h}_{\hat{x}} + \hat{u}_{\hat{z}}\Big|_{\hat{z}=\hat{h}}\right)\underline{e_1} - \delta\left(\hat{p}\hat{h}_{\hat{y}} + \hat{v}_{\hat{z}}\Big|_{\hat{z}=\hat{h}}\right)\underline{e_2} + \hat{p}\,\underline{e_3} \quad \text{as } \delta \to 0$$

The normal stress is $O(1/\delta)$ larger than the tangential stress.

As promised, we can support large loads with a small resistance to transverse motion.