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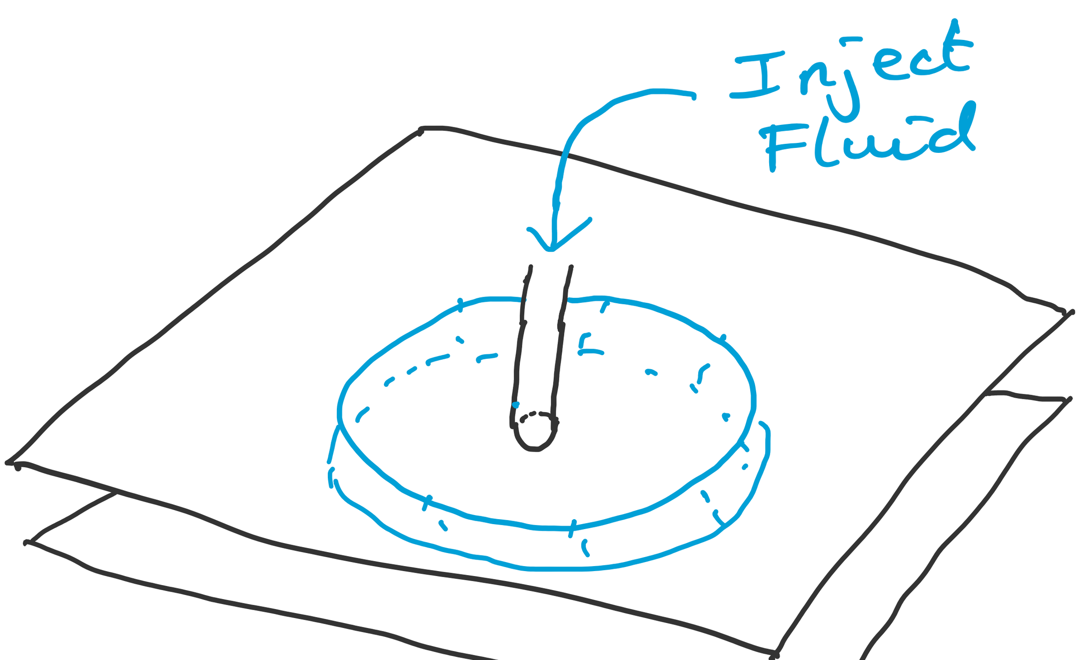
Viscous Flow Lecture 15
Flow in Hele-Shaw cells



Consider two parallel boundaries with constant spacing h, both at rest (so $U = V = 0$). Flows can be driven by (large) external pressure gradients, or by injecting or removing fluid.

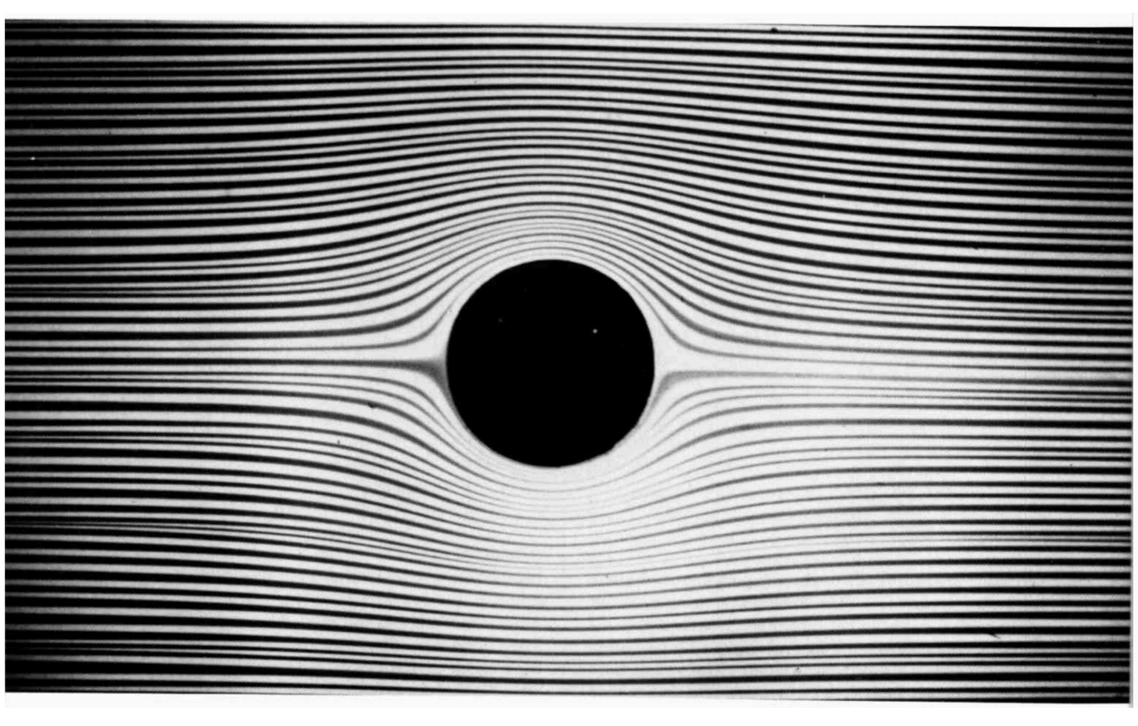
$$\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} = 0, \qquad \bar{u} = -\frac{h^2}{12}\frac{\partial p}{\partial x}$$

$$\bar{v} = -\frac{h^2}{12}\frac{\partial p}{\partial y}$$

$$\Rightarrow \quad \frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} = 0$$

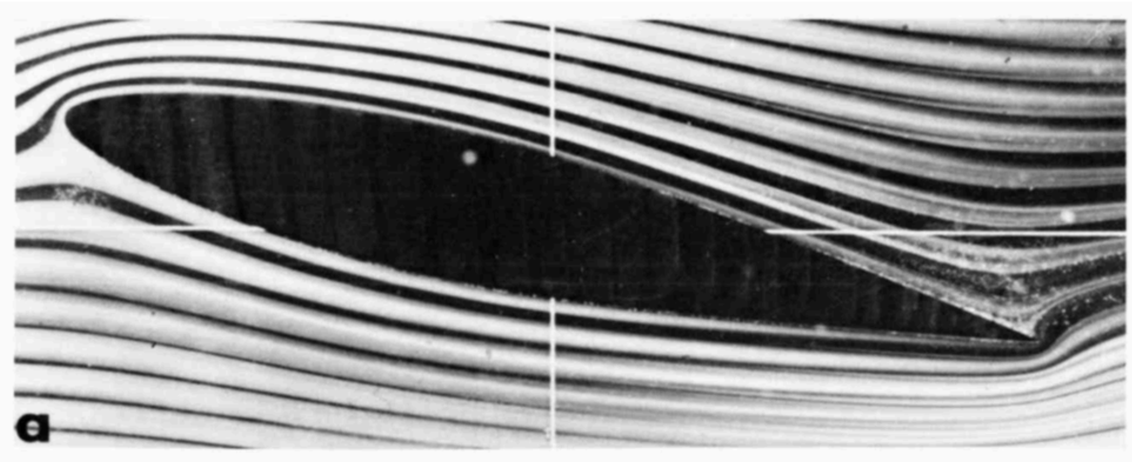
This is potential flow for $\bar{u} = \nabla\phi$ with $\phi = -\frac{h^2}{12} p$.

Flow past an object is the same as in irrotational inviscid 2D flow:



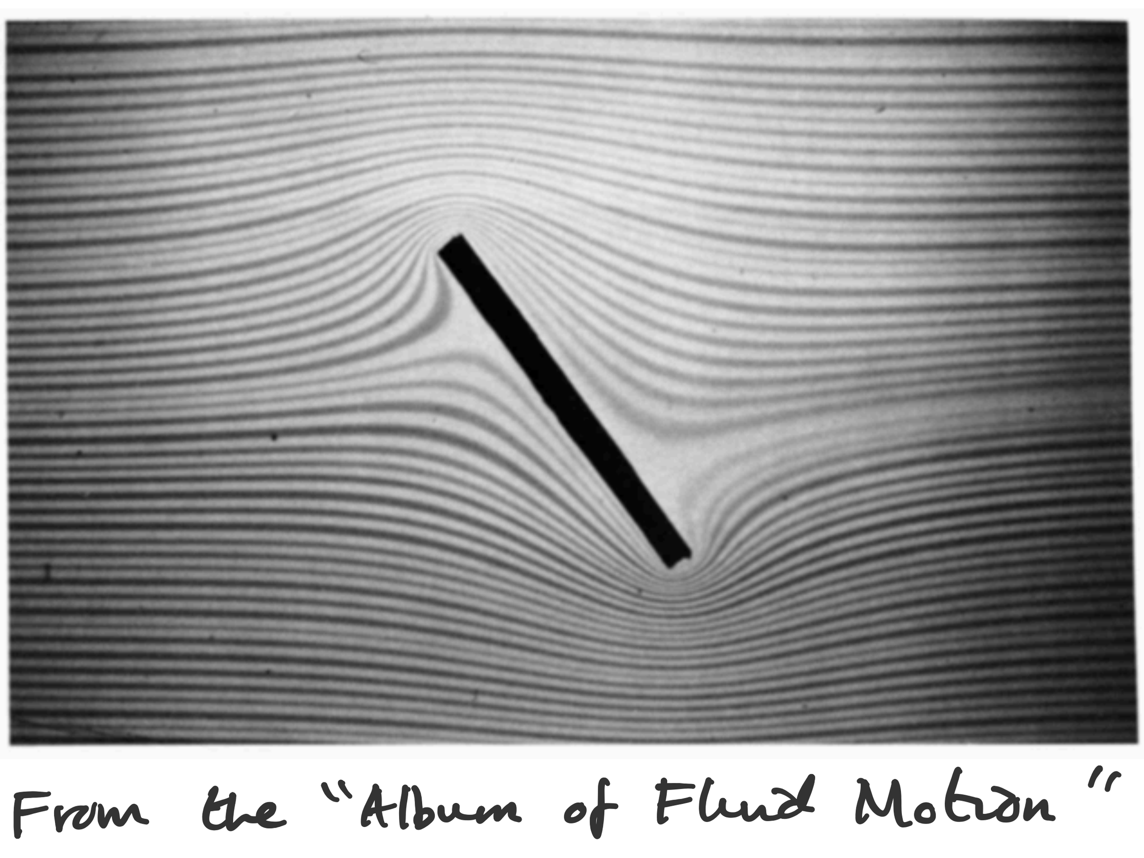
Potential flow past a cylinder

From the "Album of Fluid Motion"



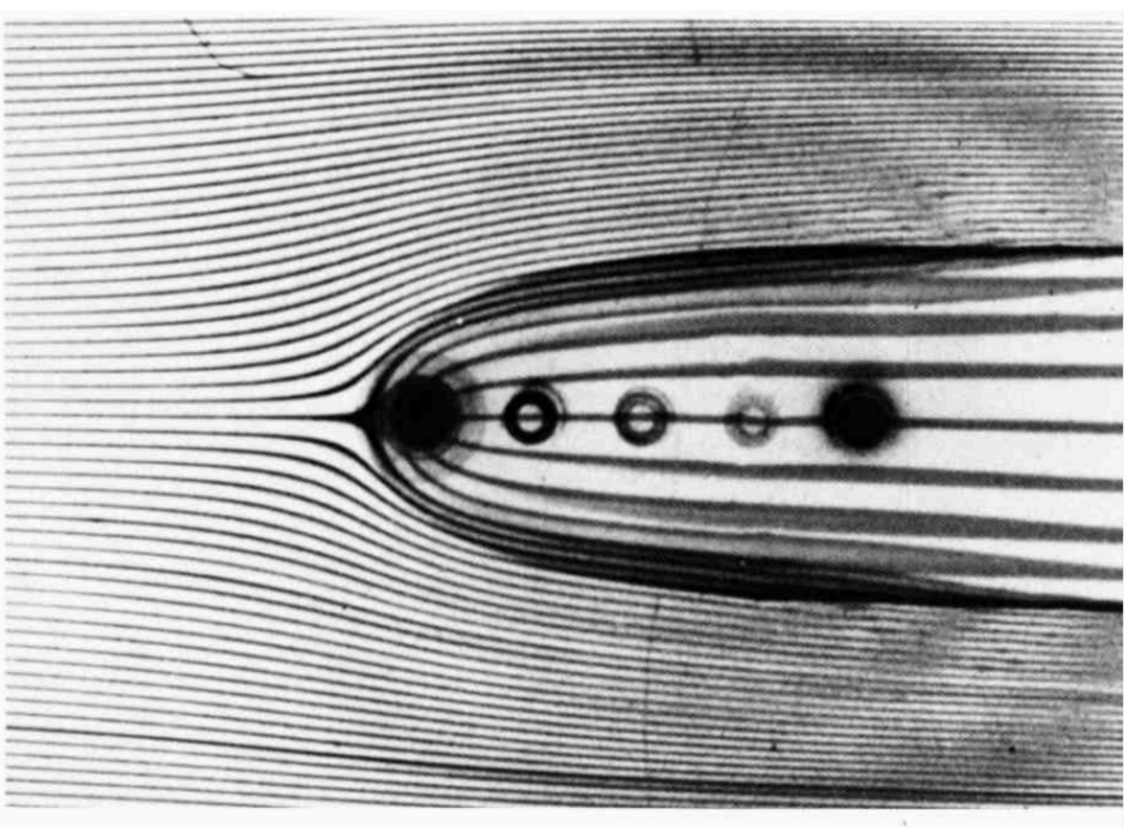
Flow past an aerofoil. Zero circulation so no Kutta condition.

From the "Album of Fluid Motion"



Flow past an inclined plate

From the "Album of Fluid Motion"



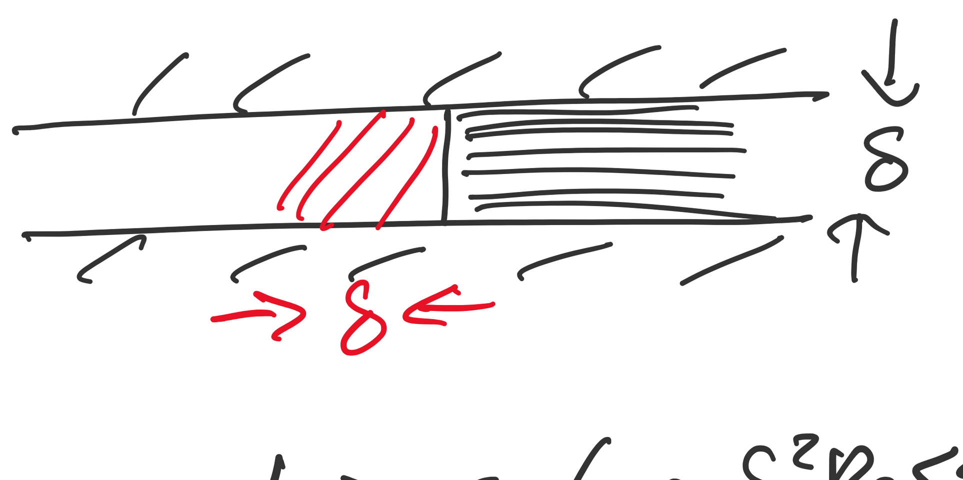
Flow past a Rankine body. Unsteady point source in a uniform stream.

From the "Album of Fluid Motion"

Streamlines are the same, but the pressure is completely different (no Bernoulli) and there can be no circulation as the pressure $p(x,y,t)$ is single-valued.

$$\Gamma = \int_C (\bar{u}, \bar{v}) \cdot d\underline{x} = 0$$ around any closed curve C.

The lubrication approximation breaks down in $O(\delta)$-thick boundary layers on smooth obstacles:



The flow is governed by the 3D slow viscous flow equations (as $\delta^2 Re \ll 1$)

The boundary layer structure is different from Prandtl's, and there is no separation.

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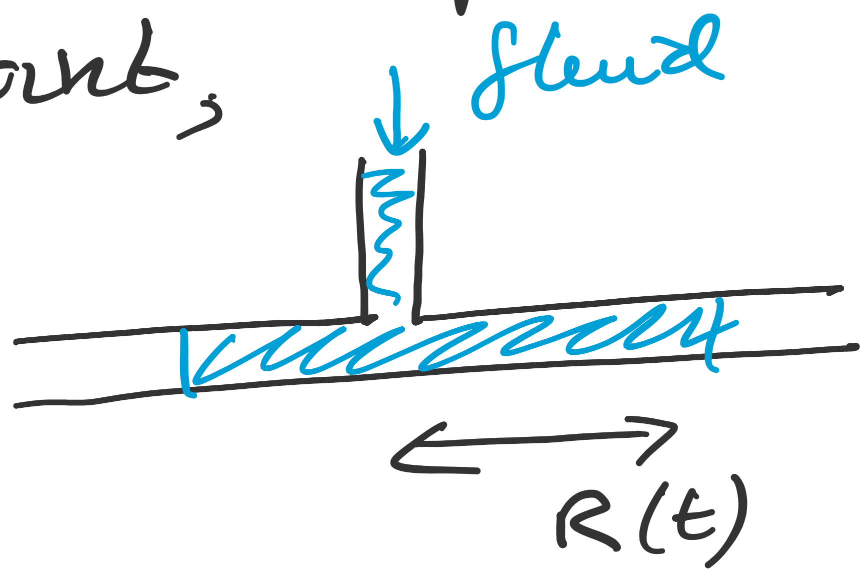
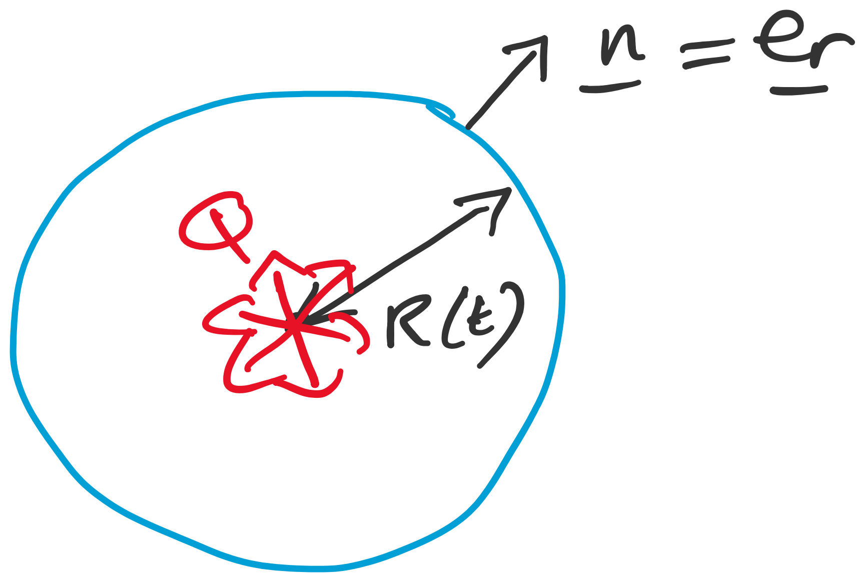
A Hele-Shaw cell with fluid injected at a point, with volume flux Q.



Free boundary problem for $p(r,t)$ and $R(t)$.



Part of the solution is finding the region occupied by the fluid.

i) $\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial p}{\partial r}\right) = 0$ in $0 < r < R$

ii) $(\bar{u},\bar{v})\cdot\underline{e_r} = -\frac{h^3}{12}\frac{\partial p}{\partial r} \sim \frac{Q}{2\pi r h}$ as $r \to 0$

iii) $(\bar{u},\bar{v})\cdot\underline{e_r} = \dot{R}$ and $p = 0$ both on $r = R$

iv) $R(0)$ given

(i) & (ii) $\Rightarrow$ $(\bar{u},\bar{v})\cdot\underline{e_r} = -\frac{h^3}{12}\frac{\partial p}{\partial r}$

$= \frac{Q}{2\pi r h}$ for $0 < r < R$

(iii) & (iv) $\Rightarrow$ $p = -\frac{6Q}{\pi h^3}\log(r/R)$

Global mass conservation

$\Rightarrow \pi h R(t)^2 = \pi h R(0)^2 + Qt$

Would the flow look the same if $Q < 0$, so we remove fluid instead of injecting fluid?

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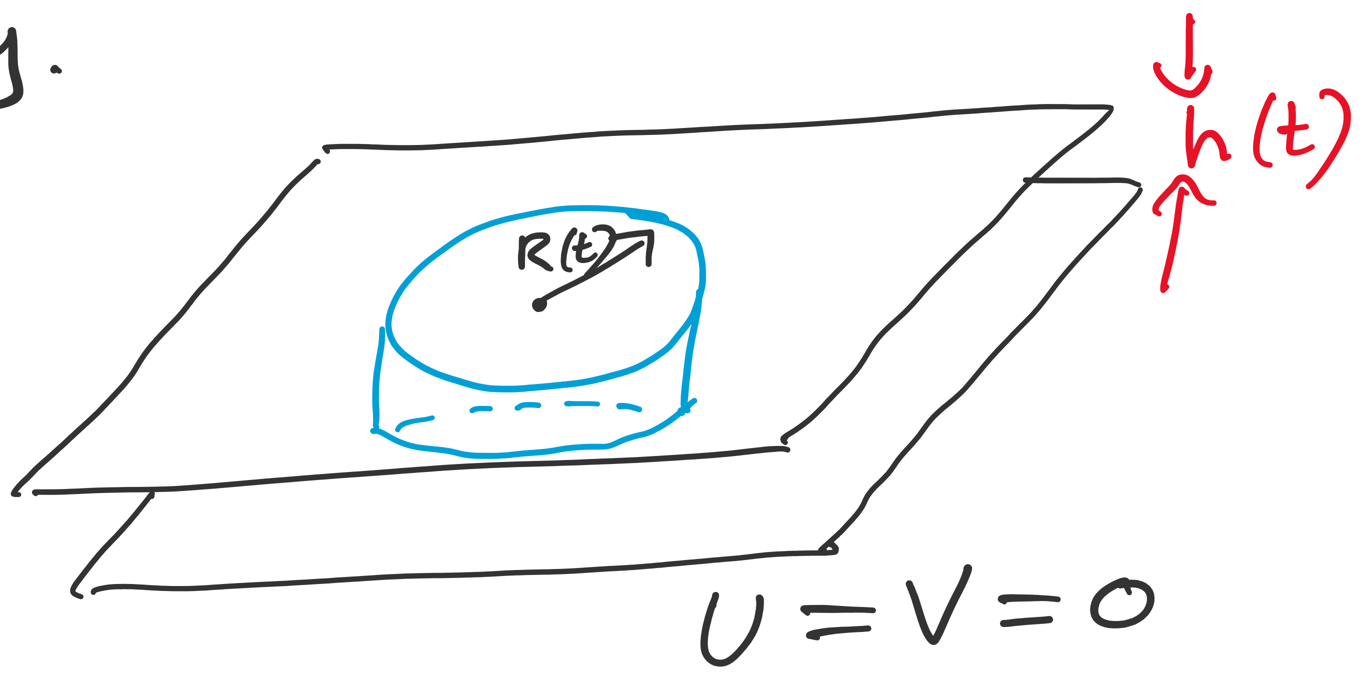
Axisymmetric squeeze film with a free boundary.

Two parallel plates, separation $h(t)$, fluid only in $r \le R(t)$.



Free boundary problem for $p(r,t)$ & $R(t)$.

i) $\frac{1}{r}\frac{\partial}{\partial r}\left(\frac{h^2}{12}\frac{\partial p}{\partial r}\right) = \dot{h}$ on $0 < r < R(t)$

ii) $(\bar{u}, \bar{v}) \cdot \underline{e_r} = -\frac{h^2}{12}\frac{\partial p}{\partial r} = \dot{R}$, $p = 0$ } on $r = R(t)$

iii) Initial radius $R(0)$ is given

i) $\Rightarrow$ $p = \frac{3\dot{h}}{h^3}\left(r^2 + A(t)\log r + B(t)\right)$

p is bounded as $r \to 0$, $p = 0$ on $r = R$, so

$$p = \frac{3\dot{h}}{h^3}\left(r^2 - R^2\right)$$

The kinematic boundary condition (ii)

$$\Rightarrow \quad \dot{R} = -\frac{h^2}{12}\left.\frac{\partial p}{\partial r}\right|_{r=R}$$

$$= -\frac{h^2}{12}\frac{3\dot{h}}{h^3}2R = -\frac{\dot{h}R}{2h}$$

$$\Rightarrow \quad 2\frac{\dot{R}}{R} + \frac{\dot{h}}{h} = 0$$

$$2\log R + \log h = \text{constant}$$

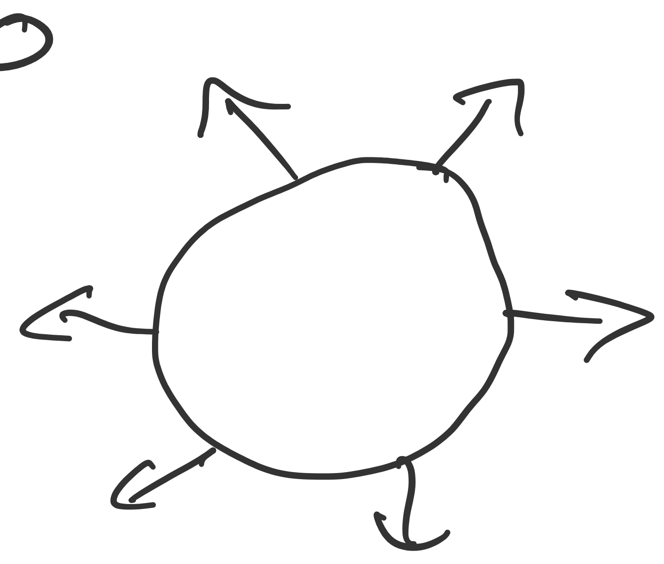
$$\pi h(t) R(t)^2 = \pi h(0) R(0)^2$$

The total volume of the cylinder of fluid is conserved.

This flow looks like it should be reversible on reversing $h(t)$.

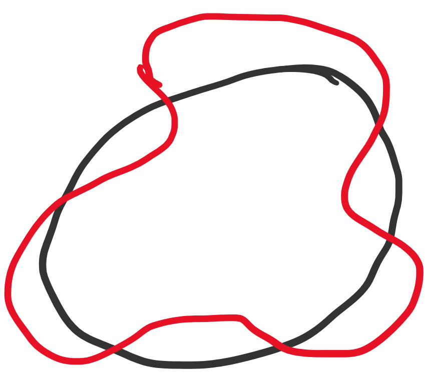
But: experimental observations

$\dot{h} < 0$



small perturbations from a circular free surface decay

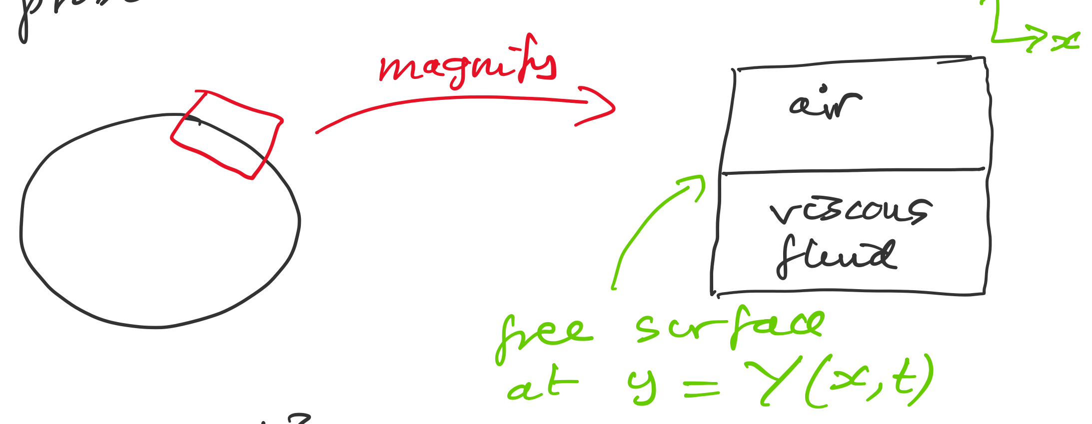
$\dot{h} > 0$



small perturbations grow into "viscous fingers."

The Saffman – Taylor instability

Zoom in to study the local problem near the free surface.



Absorb $\frac{h^2}{12}$ into the pressure. (Or call $-\phi$ the "pressure")

We could also consider linearising for small changes in h over time.

Flow is now governed by

$$\bar{u}_x + \bar{v}_y = 0, \quad \left.\begin{aligned}\bar{u} &= -p_x,\\ \bar{v} &= -p_y\end{aligned}\right\} \text{all in } y < Y$$

Zero-pressure and kinematic boundary conditions:

$$p = 0, \quad \bar{v} = Y_t + \bar{u}\,Y_x \quad \text{on } y = Y$$

The base state (with no instability) is a steady travelling wave solution moving with speed $\mathcal{V}$, $\bar{u} = 0$, $\bar{v} = \mathcal{V}$, $Y = \mathcal{V}t$ if we choose the time origin correctly.

$$p = p_0(y,t) = -\mathcal{V}(y - \mathcal{V}t)$$

To drive the flow (treating h as fixed) we prescribe the pressure gradient $\frac{\partial p}{\partial y} \to -\mathcal{V}$ as $y \to -\infty$.

Key step: change to travelling wave coordinates (x, η, t) with $\eta = y - \mathcal{V}t$ and set:

$$\bar{u} = \varepsilon\,\hat{u}(x,\eta,t)$$
$$\bar{v} = \mathcal{V} + \varepsilon\,\hat{v}(x,\eta,t)$$
$$p = p_0(y,t) + \varepsilon\,\hat{p}(x,\eta,t)$$
$$Y - \mathcal{V}t = \varepsilon\,\hat{Y}(x,t)$$

ε is a small parameter characterising the size of the initial perturbation.

Using the chain rule gives

$$\hat{u}_x + \hat{v}_\eta = 0, \quad \hat{u} = -\hat{p}_x, \quad \hat{v} = -\hat{p}_\eta$$

in $\eta < \varepsilon\hat{Y}$, with boundary conditions

$$\left.\begin{aligned}\hat{p} &= \mathcal{V}\,\hat{Y}\\ \hat{v} &= \hat{Y}_t + \varepsilon\,\hat{u}\,\hat{Y}_x\end{aligned}\right\} \text{both on } \eta = \varepsilon\hat{Y},$$

$$\hat{p} = o(-\eta) \quad \text{as} \quad \eta \to -\infty.$$

asymptotically smaller than $|\eta|$ as $\eta \to -\infty$.

Step i) As in the Part A Fluids treatment of water waves, linearise and impose the boundary conditions on the unperturbed interface at $\eta = 0$.

$$\hat{u}_x + \hat{v}_\eta = 0, \quad \hat{u} = -\hat{p}_x, \quad \hat{v} = -\hat{p}_\eta$$

all in $\eta < 0$,

$$\hat{p} = \mathcal{V}\,\hat{Y} \quad \text{and} \quad \hat{v} = \hat{Y}_t \quad \text{on } \eta = 0,$$

$$\hat{p} = o(-\eta) \quad \text{as} \quad \eta \to -\infty$$

Step ii) Seek a separable solution with wavenumber k and complex growth rate λ:

$$\hat{p} = g(\eta)\,e^{ikx + \lambda t}, \quad \hat{Y} = e^{ikx + \lambda t}$$

$$\Rightarrow \quad g'' - k^2 g = 0 \quad \text{in} \quad \eta < 0$$

$$g(0) = \mathcal{V} \quad \text{and} \quad g'(0) = -\lambda$$

$$g'(\eta) \text{ bounded as } \eta \to -\infty.$$

Step iii) This is an eigenvalue problem for λ as a function of k.

$$g(\eta) = A\,e^{|k|\eta}$$

satisfies ODE $g'' - k^2 g = 0$, decays as $\eta \to -\infty$

BCs at $\eta = 0 \Rightarrow A = \mathcal{V}$
$$\lambda = -A|k|$$

$$\therefore \quad \boxed{\lambda = -\mathcal{V}\,|k|}$$

Hence perturbations decay ($\lambda < 0$) if $\mathcal{V} > 0$.

Perturbations grow ($\lambda > 0$) if $\mathcal{V} < 0$.

This is an example of the Saffman – Taylor instability: the interface between two immiscible fluids is unstable when it moves towards the more viscous fluid.

The growth rate of the instability is proportional to $|k|$. Shorter wavelength (larger $|k|$) perturbations grow faster into "viscous fingers" whose evolution is very sensitive to the driving force, surface tension, roughness of the boundaries etc.