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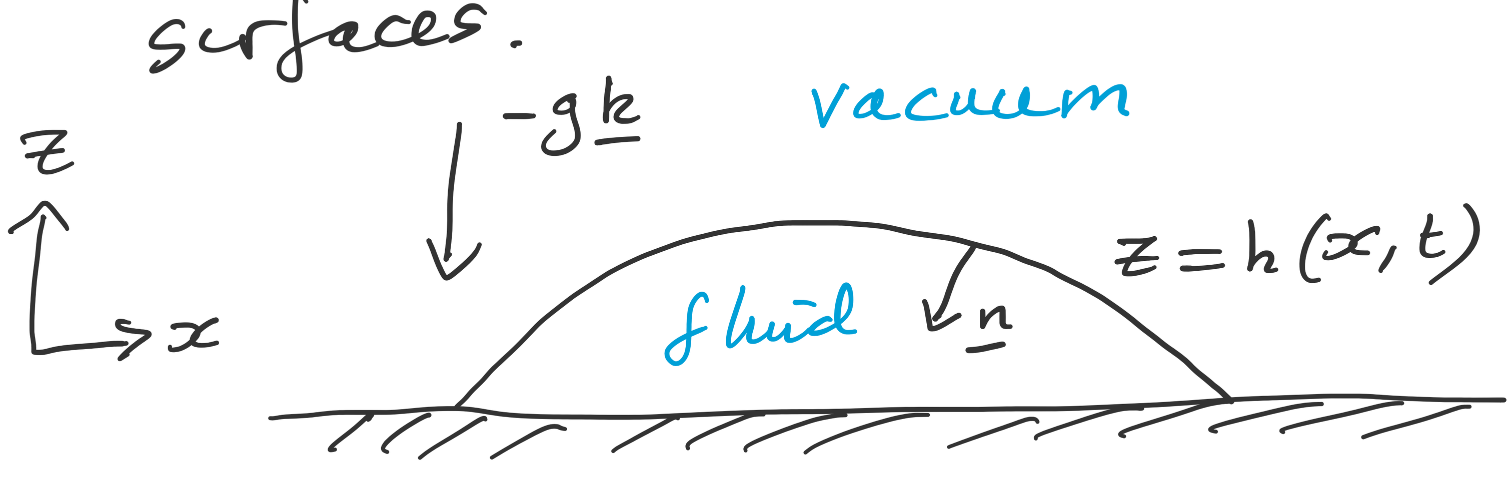
Viscous Flow Lecture 16

Thin Films: Lubrication theory for gravity-driven flows with free surfaces.



Simple model for spreading raindrops, molten lava, or ice sheets.

The fluid domain has a free boundary at $z = h(x,t)$ in 2D.

Dimensional problem

Incompressible Navier–Stokes equations:

$$\nabla\cdot\underline{u} = 0, \quad \rho\frac{D\underline{u}}{Dt} = -\nabla p + \mu\nabla^2\underline{u} - \rho g\underline{k}$$

where $\underline{u}(x,z,t) = u\underline{i} + w\underline{k}$

$p(x,z,t)$

No-flux and no-slip on the bottom boundary: $u = w = 0$ on $z = 0$.

We need to impose 3 boundary conditions on the free surface, one more is needed to determine its location.

Kinematic BC (no-flux)

$$w = h_t + u h_x \quad \text{on } y = h$$

From considering $\frac{D}{Dt}(h(x,t) - z) = 0$

Dynamic BCs

No-stress: $\underline{\underline{\sigma}}\cdot\underline{n} = 0$ on $y = h$

This has 2 components:

$\underline{n}\cdot\underline{\underline{\sigma}}\cdot\underline{n} = 0$ (pressure)

$\underline{t}\cdot\underline{\underline{\sigma}}\cdot\underline{n} = 0$ (viscous stress)

The unit inward (into the fluid) pointing normal vector is

$$\underline{n} = \frac{(\underline{i}h_x - \underline{k})}{\sqrt{1 + h_x^2}}.$$

$\underline{t}$ is tangential to the boundary, so perpendicular to $\underline{n}$.

Dimensionless problem

Scale as before for lubrication theory:

$$x = L\hat{x}, \quad z = \delta L\hat{z}, \quad h = \delta L\hat{h}$$

$$u = U\hat{u}, \quad w = \delta U\hat{w}, \quad t = \frac{L}{U}\hat{t}$$

$$p = \frac{\mu U}{\delta^2 L}\hat{p}$$

We're taking $p_{atm} = 0$, consistent with vacuum above the fluid.

The velocity scale U must be determined. No longer set by moving boundaries.

$$\delta^2 Re\frac{D\hat{u}}{D\hat{t}} = -\hat{p}_{\hat{x}} + \delta^2\hat{u}_{\hat{x}\hat{x}} + \hat{u}_{\hat{z}\hat{z}}$$

$$\delta^4 Re\frac{D\hat{w}}{D\hat{t}} = -\hat{p}_{\hat{z}} + \delta^4\hat{w}_{\hat{x}\hat{x}} + \delta^2\hat{w}_{\hat{z}\hat{z}} - \frac{\delta^3\rho g L^2}{\mu U}$$

$$\hat{u}_{\hat{x}} + \hat{w}_{\hat{z}} = 0.$$

Choose the velocity scale U to get a flow driven by gravity:

$$U = \frac{\delta^3\rho g L^2}{\mu} = \frac{\delta^3 g L^2}{\nu}$$

Assume $\delta \ll 1$ and $\delta^2 Re = \frac{\delta^5 g L^3}{\nu^2} \ll 1$

At leading order we obtain the lubrication equations

$$\hat{u}_{\hat{z}\hat{z}} = \hat{p}_{\hat{x}}, \quad \hat{p}_{\hat{z}} = -1$$ New term from gravity

$$\hat{u}_{\hat{x}} + \hat{w}_{\hat{z}} = 0$$

BCs on $\hat{z} = 0$: $\hat{u} = \hat{w} = 0$.

Boundary conditions on the free surface:

$$\underline{\underline{\sigma}}\cdot\underline{n} \sim -\frac{\mu U}{\delta L}\left(\hat{p}\hat{h}_{\hat{x}} + \hat{u}_{\hat{z}}\Big|_{\hat{z}=\hat{h}}\right)\underline{i}$$

Dimensional normal stress

$$+ \frac{\mu U}{\delta^2 L}\hat{p}\,\underline{k} \quad \text{as } \delta\to 0.$$

The dimensionless BCs on $\hat{z} = \hat{h}$ are

$\hat{w} = \hat{h}_{\hat{t}} + \hat{u}\hat{h}_{\hat{x}}$ kinematic BC

$\hat{p} = 0$ zero normal stress

$\hat{u}_{\hat{z}} = 0$ zero tangential stress

Hence $\hat{p} = \hat{h} - \hat{z}$ is the hydrostatic pressure in the fluid due to the weight of the fluid above. It comes from $\frac{\partial p}{\partial z} = -\rho g$ in dimensional variables.

Also, $\hat{u} = \hat{z}\left(\frac{1}{2}\hat{z} - \hat{h}\right)\hat{h}_{\hat{x}}$.

The zero tangential stress BC gives us half a parabolic profile.

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Layer-integrated equations

The layer-integrated incompressibility condition is

$$0 = \int_0^{\hat h} \hat u_{\hat x} + \hat w_{\hat z} \, d\hat z$$

$$= \frac{\partial}{\partial \hat x}\left(\int_0^{\hat h} \hat u \, d\hat z\right) - \underbrace{\hat u \hat h_{\hat x}\Big|_{z=\hat h} + [\hat w]_0^{\hat h}}_{= \hat h_{\hat t}}$$

where $\frac{\partial}{\partial \hat x}\left(\int_0^{\hat h} \hat u \, d\hat z\right)$ has $= -\frac{1}{3}\hat h^3 \hat h_{\hat x}$ written under it.

$$\therefore \quad \hat h_{\hat t} = \frac{\partial}{\partial \hat x}\left(\frac{1}{3}\hat h^3 \frac{\partial \hat h}{\partial \hat x}\right) = \frac{\partial^2}{\partial \hat x^2}\left(\frac{\hat h^4}{12}\right)$$

This is a closed equation for evolving $\hat h(\hat x, \hat t)$.

Now drop hats for simplicity.

This is a nonlinear diffusion equation

$$\partial_t h = \partial_x (D(h)\partial_x h)$$

with a diffusivity or "mobility" $D(h) = \frac{1}{3}h^3$ that vanishes at $h=0$.

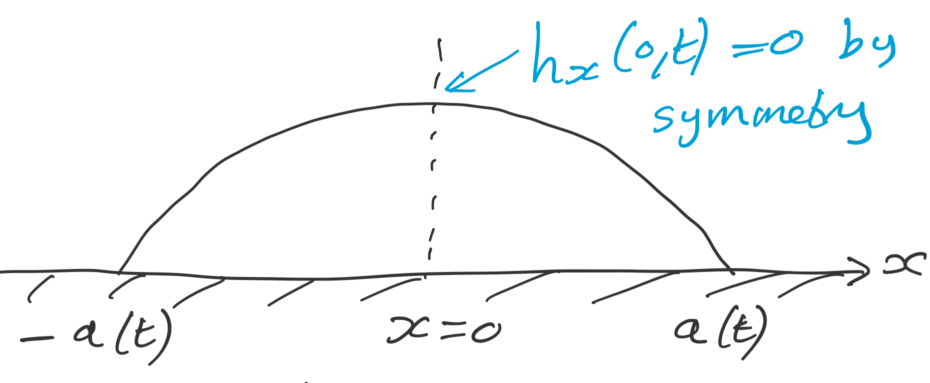
$\partial_t h = \partial_x(\text{flux})$

with a $\text{flux} = D(h)\partial_x h$.

This equation has compactly-supported similarity solutions (i.e. $h \equiv 0$ outside some interval for x)

E.g. consider a spreading drop with support $[-a(t), a(t)]$, so $h(\pm a(t), t) = 0$.

Impose initial conditions for h and a so that $\int_{-a(0)}^{a(0)} h(x,0)\,dx = M$, say.



Seek a similarity solution:

$$h(x,t) = t^n f(\eta) \quad \text{where } \eta = \frac{x}{t^m}$$

$$a = A t^m$$

for two constants m and n to be determined.

$$\frac{\partial \eta}{\partial x} = \frac{1}{t^m}, \quad \frac{\partial \eta}{\partial t} = -\frac{mx}{t^{m+1}} = -\frac{m}{t}\eta$$

$$\therefore \quad \frac{\partial h}{\partial t} = n t^{n-1} f(\eta) + t^n f'(\eta)\frac{\partial \eta}{\partial t}$$

$$= t^{n-1}(nf - m\eta f')$$

$$\frac{\partial}{\partial x}(h^4) = t^{4n}\frac{\partial}{\partial x}(f^4)$$

$$= t^{4n}(f^4)'\frac{\partial \eta}{\partial x}$$

$$= t^{4n-m}(f^4)'$$

$'$ means $\frac{\partial}{\partial \eta}$

$$\frac{\partial^2}{\partial x^2}(h^4) = t^{4n-2m}(f^4)''$$

$$\therefore \quad \frac{\partial h}{\partial t} = \frac{\partial^2}{\partial x^2}\left(\frac{h^4}{12}\right) \text{ becomes}$$

$$t^{n-1}(nf - m\eta f') = t^{4n-2m}(f^4)''$$

t cancels if $n-1 = 4n-2m$, leaving an ODE in η for f.

Mass conservation requires

$$M = \int_{-a(t)}^{a(t)} h(x,t)\,dx$$

$$= \int_{-A}^{A} t^n f(\eta) t^m \, d\eta$$

$$= t^{n+m}\int_{-A}^{A} f(\eta)\,d\eta.$$

This is only independent of t if $n+m=0$.

Hence $n = -1/5$ and $m = 1/5$.

The ODE becomes

$$(f^4)'' = -\frac{12}{5}(f + \eta f')$$

$$= -\frac{12}{5}(\eta f)'$$

Integrate: $(f^4)' = 4f^3 f' = -\frac{12}{5}\eta f + B$

where B is an integration constant.

The flux is zero at either $\eta = 0$, symmetry condition, or at $f=0$. Both are satisfied when $B=0$.

We know $f(\eta) > 0$ for $0 \le \eta < A$

so

$$f^2 f' = -\frac{3}{5}\eta.$$

$$\left(\frac{1}{3}f^3\right)' = -\frac{3}{5}\eta$$

$$f^3 = \frac{9}{10}(A^2 - \eta^2)$$

After choosing a constant to make $f=0$ at $\eta = \pm A$.

$$\therefore \quad f(\eta) = \left[\frac{9}{10}(A^2-\eta^2)\right]^{1/3}$$

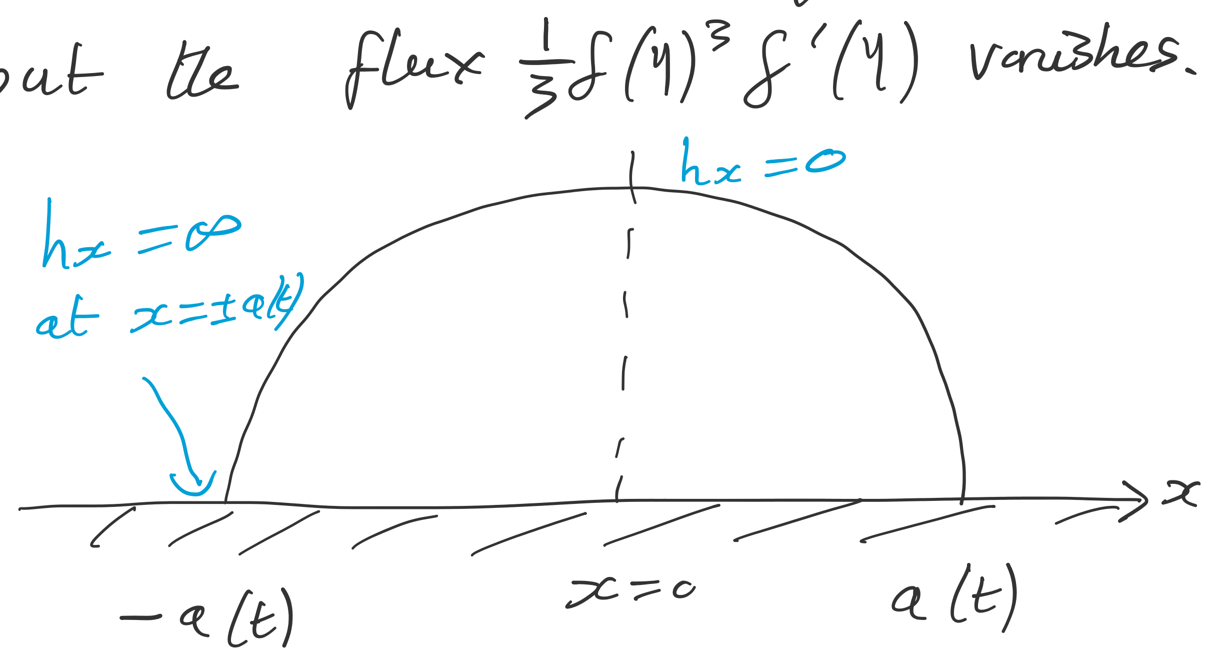
where A is determined by

$$M = \int_{-A}^{A} f(\eta)\,d\eta.$$

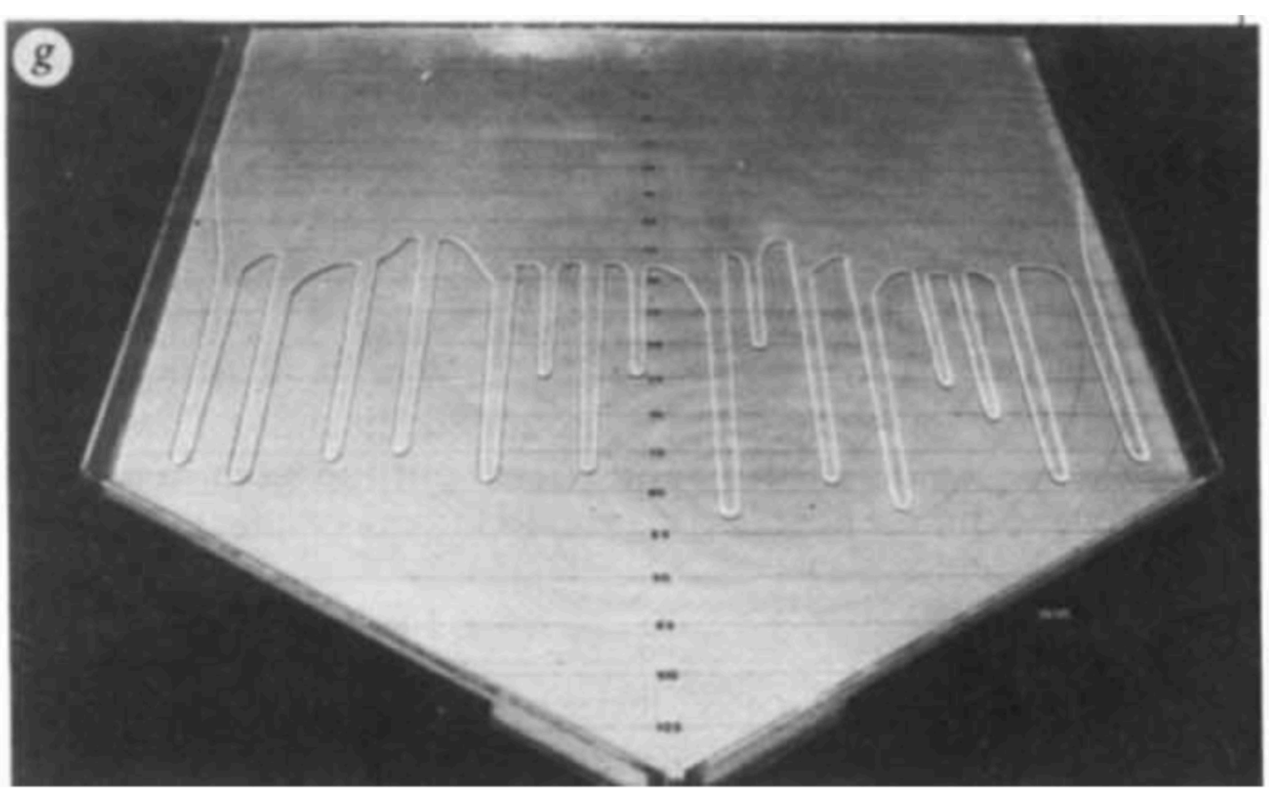
The derivative

$$f'(\eta) = -\frac{3}{5}\eta\left[\frac{9}{10}(A^2-\eta^2)\right]^{-2/3}$$

becomes infinite as $\eta \to \pm A$, but the flux $\frac{1}{3}f(\eta)^3 f'(\eta)$ vanishes.



Thin films on an inclined surface:



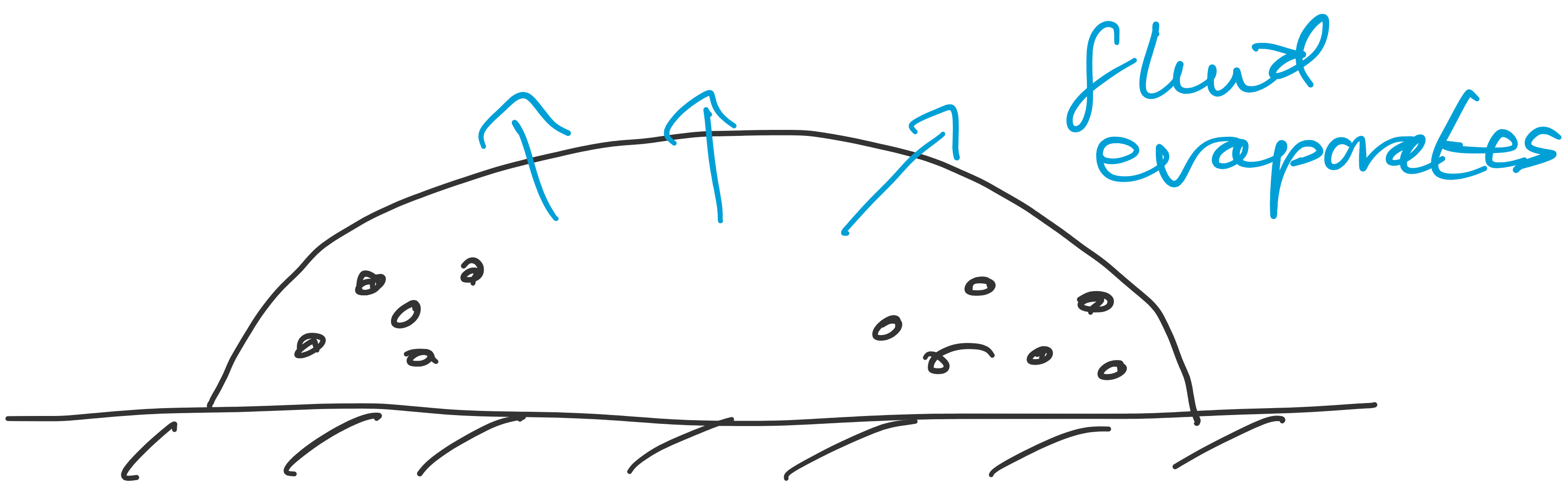
From Huppert (1982) Nature

A uniform interface breaks up into "viscous fingers" but here in a way that depends on surface tension.

More on low Reynolds number flows in the last 6 lectures of M Math Phys "Advanced Fluid Dynamics" for suspensions of lots of spheres in Stokes flow.
Behaves like a viscous fluid with Einstein's viscosity

$$\mu = \mu_{fluid}\left(1 + \tfrac{5}{2}\phi\right)$$

where ϕ is the volume fraction occupied by the spheres.



Particles in the spreading drop flow.
Particles concentrate at the leading edge, creating "coffee rings".

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