Questions marked as optional are not expected to be handed in and will not be marked.

1. Find a parametric solution for the PDE

$$u_x + uu_y = 1,$$

with $u = x/2$ on $y = x$, $0 \le x \le 1$, and state the domain of definition.

2. Find the solution of

$$yu_x - 2xyu_y = 2xu$$

such that $u = y^3$ when $x = 0$ and $1 \le y \le 2$. What is the domain of validity of the solution? Describe the behaviour of u as $y \to 0+$ in this domain.

3. Find, in parametric form, the solution of

$$(x-u)u_x + u_y + u = 0$$

with $u = 1$ on $y = x$, $0 < x < 1/2$. Show that u is determined in the region

$$-\sinh y < x < \frac{\mathrm{e}^{1/2-y}}{2}.$$

4. (a) Suppose the ODEs

$$\frac{\mathrm{d}x}{a(x,y,u)} = \frac{\mathrm{d}y}{b(x,y,u)} = \frac{\mathrm{d}u}{c(x,y,u)}$$

have two linearly independent solutions $f(x,y,u) = \text{const}$ and $g(x,y,u) = \text{const}$. Explain why f and g must satisfy the equations

$$af_x + bf_y + cf_u = 0, \qquad ag_x + bg_y + cg_u = 0.$$

(b) *[This part is optional, but its application to the example in (c).]*
Show that, if $u(x,y)$ is determined implicitly by the relation

$$f(x,y,u) = F\big(g(x,y,u)\big),$$

where F is any (suitably smooth) function, then $u(x,y)$ satisfies the PDE

$$a\frac{\partial u}{\partial x} + b\frac{\partial u}{\partial y} = c.$$

[*Hint: Differentiate with respect to x and y and then try to eliminate f_x, f_y, g_x, g_y.*]

(c) Hence show that the general solution of the PDE $yu_x + u^2u_y = u^2$ is given by

$$u = y + F\left(x + \frac{y}{u} - \log u\right).$$

5. Find the explicit solution of the PDE

$$(1+u)\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = u$$

subject to the boundary data

(*i*) $u(x,1) = x$ for $0 \le x \le 1$; (*ii*) $u(x,1) = -x$ for $0 \le x \le 1$.

In case (i), state where the solution is uniquely determined and sketch this region in the (x, y)-plane.

In case (ii), show that all the characteristic projections pass through the point $(x, y) = (\log 2, 2)$, where the Jacobian $J = |\partial(x, y)/\partial(\tau, s)| = 0$. Hence find and sketch the region of the (x, y)-plane where u is uniquely determined. Explain what happens to the graph of $u(x, y)$ versus x as y increases from 1 to 2.

6. Solve the PDE

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(\frac{u^2}{2}\right) = 0$$

for $t > 0$, subject to $u(x, 0) = f(x)$, in each of the following two cases:

$$(i) \quad f(x) = \begin{cases} 0 & x < 0, \\ x & 0 \le x < 1, \\ 1 & 1 \le x, \end{cases} \qquad\qquad (ii) \quad f(x) = \begin{cases} 0 & x < 0, \\ -x & 0 \le x < 1, \\ -1 & 1 \le x. \end{cases}$$

In case (ii), find a single-valued weak solution by introducing a shock. Sketch the resulting solution $u(x, t)$ versus x as t varies, and the characteristic projections in the (x, t)-plane.

7. Solve the Cauchy problem

$$\begin{aligned} \frac{\partial u}{\partial x} + u\frac{\partial u}{\partial y} &= 0 & \qquad x > 0, \\ u &= u_0(y) & \qquad x = 0, \end{aligned}$$

where

$$u_0(y) = \begin{cases} y(1-y) & 0 < y < 1, \\ 0 & y < 0, \ y > 1. \end{cases}$$

Show that u becomes multi-valued on the curve $y = (1 + x)^2/(4x)$, $x > 1$. Sketch the characteristic projections in the (x, y)-plane and indicate where a unique classical solution exists. Also sketch profiles of $u(x, y)$ versus y for different values of $x > 0$.

8. Suppose $u(x, t)$ satisfies

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \epsilon\frac{\partial^2 u}{\partial x^2}, \qquad\qquad 0 < \epsilon \ll 1.$$

Show that the change of variables to ξ and τ, where $t = \tau$, $x - Vt = \epsilon\xi$, gives

$$(-V + u)\frac{\partial u}{\partial \xi} = \frac{\partial^2 u}{\partial \xi^2} \tag{1}$$

when small terms of order ϵ are neglected. Assuming that $u \to u_\pm$ as $\xi \to \pm\infty$ (where $u\pm$ are constants), deduce that

$$V = \tfrac{1}{2}\left(u_- + u_+\right), \qquad\qquad u_+ \le u_-,$$

Determine the form of and sketch the solution $u(\xi)$, and use this to interpret the nature and location of shocks in the system when $\epsilon = 0$.

Hint: to show $u_+ \le u_-$, first multiply (1) by u and integrate to obtain the relation

$$\left[-\frac{Vu^2}{2} + \frac{u^3}{3}\right]_{-\infty}^{+\infty} + \int_{-\infty}^{\infty}\left(\frac{\partial u}{\partial \xi}\right)^2 \mathrm{d}\xi = 0.$$