1. In this question all functions and distributions are assumed real-valued.

(a) [10 marks] What does it mean to say that a distribution $u \in \mathscr{D}'(\mathbb{R}^n)$ has order $m \in \mathbb{N}_0$? Show that a positive distribution $v \in \mathscr{D}'(\mathbb{R}^n)$ has order 0 and explain in what sense it is a measure.

If u, $v \in \mathscr{D}'(\mathbb{R}^n)$ are two distributions we write $u \leqslant v$ when $v - u \geqslant 0$, that is, when $v - u$ is a positive distribution.

Show that if $u \leqslant v$, then u has order $m \in \mathbb{N}_0$ if and only if v has order $m \in \mathbb{N}_0$.

What can you say about a distribution $w \in \mathscr{D}'(\mathbb{R})$ that satisfies $w' \geqslant 0$?

(b) [8 marks] In each of the following cases find all the distributions $u \in \mathscr{D}'(\mathbb{R})$ that satisfies the given differential inequality.

(1)
$$u' \geqslant 1$$

(2)
$$u' \geqslant u$$

(3)
$$u'' \geqslant u$$

(c) [7 marks] Denote $\operatorname{sgn}(t) = t/|t|$ for $t \in \mathbb{R} \setminus \{0\}$ and $\operatorname{sgn}(0) = 0$. Define for each $t > 0$ the function $T_t(y) = \sqrt{y^2 + t}$, $y \in \mathbb{R}$.

(i) Assume $f \in \mathrm{C}^\infty(\mathbb{R}^n)$. Calculate $T_t(f)\partial_j T_t(f)$ for each j and deduce the formula

$$\left|\nabla T_t(f)\right|^2 + T_t(f)\Delta T_t(f) = \left|\nabla f\right|^2 + f\Delta f.$$

Use this to conclude that

$$\Delta T_t(f) \geqslant \frac{f}{T_t(f)}\Delta f$$

holds on $\mathbb{R}^n$.

(ii) Assume $u \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^n)$ and that $\Delta u \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^n)$. Prove, for instance using mollifiers, that

$$\Delta|u| \geqslant \operatorname{sgn}(u)\Delta u \quad \text{in } \mathscr{D}'(\mathbb{R}^n).$$