Distribution Theory MT20

Problem Sheet 2

Problem 1. Let $f, g \in \mathrm{C}^1(\mathbb{R})$ and define

$$u(x) = \begin{cases} f(x) & \text{if } x < 0 \\ g(x) & \text{if } x \geq 0. \end{cases}$$

Explain why $u \in \mathscr{D}'(\mathbb{R})$ and calculate the distributional derivative u'. What can you say about the function

$$v(x) = \begin{cases} f(x) & \text{if } x < 0 \\ a & \text{if } x = 0 \\ g(x) & \text{if } x > 0, \end{cases}$$

where $a \in \mathbb{R}$ is a constant that is different from both $f(0)$ and $g(0)$?

Problem 2. (a) Prove that if $f \colon \mathbb{R} \to \mathbb{R}$ is piecewise continuous and $k \in \mathbb{R}$, then the function $u(x,t) = f(x-kt)$, $(x,t) \in \mathbb{R}^2$, is locally integrable on $\mathbb{R}^2$. Conclude that it defines a distribution and show that it satisfies the one-dimensional wave equation:

$$\frac{\partial^2 u}{\partial t^2} = k^2 \frac{\partial^2 u}{\partial x^2}$$

in the sense of distributions on $\mathbb{R}^2$.
(b) Prove that $u(x,y) = \log\big(x^2+y^2\big)$ is locally integrable on $\mathbb{R}^2$, and that we have

$$\Delta u = 4\pi\delta_0$$

in the sense of distributions on $\mathbb{R}^2$, where δ_0 is the Dirac delta function on $\mathbb{R}^2$ concentrated at the origin.

Problem 3. Let $a > 0$. For each $\varphi \in \mathscr{D}(\mathbb{R})$ we let

$$\langle T_a, \varphi \rangle = \left(\int_{-\infty}^{-a} + \int_{a}^{\infty} \right) \frac{\varphi(x)}{|x|}\,\mathrm{d}x + \int_{-a}^{a} \frac{\varphi(x)-\varphi(0)}{|x|}\,\mathrm{d}x.$$

Show that T_a hereby is well-defined and that it is a distribution on $\mathbb{R}$.

Now assume that $\varphi \in \mathscr{D}(\mathbb{R})$ satisfies $\varphi(0) = 0$. Show that then

$$\langle T_a, \varphi \rangle = \int_{-\infty}^{\infty} \frac{\varphi(x)}{|x|}\,\mathrm{d}x.$$

What distribution is $T_a - T_b$ for $0 < b < a$?

Problem 4. (a) Let $\alpha \in (-n, \infty)$ and $u_\alpha(x) = |x|^\alpha$ for $x \in \mathbb{R}^n \setminus \{0\}$. Show that u_α is a regular distribution on $\mathbb{R}^n$. (*Hint: Use polar coordinates.*)

(b) For each $r > 0$ we define the r-dilation of a test function $\varphi \in \mathscr{D}(\mathbb{R}^n)$ by the rule

$$(d_r\varphi)(x) = \varphi(rx), \quad x \in \mathbb{R}^n.$$

Extend the r-dilation to distributions $u \in \mathscr{D}'(\mathbb{R}^n)$.
(c) Show that for the distribution u_α defined in (a) we have $d_r u_\alpha = r^\alpha u_\alpha$ for all $r > 0$. We express this by saying that u_α is *homogeneous of degree* α.
(d) Show that the Dirac delta function δ_0 concentrated at the origin $0 \in \mathbb{R}^n$ is homogeneous of degree $-n$.
(e) Let $u \in \mathscr{D}'(\mathbb{R}^n)$ be homogeneous of degree $\beta \in \mathbb{R}$: $d_r u = r^\beta u$ for all $r > 0$. Show that for each $j \in \{1, \ldots, n\}$ the distribution $x_j u$ is homogeneous of degree $\beta + 1$ and that the distribution $D_j u$ is homogeneous of degree $\beta - 1$. Finally show that

$$\sum_{j=1}^{n} x_j D_j u = \beta u. \tag{1}$$

This PDE is known as Euler's relation for β-homogeneous distributions.
(f) (Optional) Show that a distribution $u \in \mathscr{D}'(\mathbb{R}^n)$ that satisfies (1) must be homogeneous of degree β.

Problem 5. Show that δ_a, the Dirac delta function concentrated at $a \in \mathbb{R}$, satisfies the equation

$$(x - a)u = 0. \tag{2}$$

Find the general solution $u \in \mathscr{D}'(\mathbb{R})$ to (2). (*Hint: See Corollary 1.10 in the Lecture Notes.*)

Problem 6. (Distribution defined by principal value integral)
Define for each $\varphi \in \mathscr{D}(\mathbb{R})$,

$$\langle \mathrm{pv}\big(\frac{1}{x}\big), \varphi \rangle = \lim_{a \to 0^+} \left(\int_{-\infty}^{-a} + \int_{a}^{\infty} \right) \frac{\varphi(x)}{x} \, \mathrm{d}x.$$

(a) Show that hereby $\mathrm{pv}\big(\frac{1}{x}\big) \in \mathscr{D}'(\mathbb{R})$ and that it is homogeneous of order -1 (see Problem 4). Check that

$$\frac{\mathrm{d}}{\mathrm{d}x} \log|x| = \mathrm{pv}\big(\frac{1}{x}\big).$$

(b) Show that $u = \mathrm{pv}\big(\frac{1}{x}\big)$ solves the equation

$$xu = 1 \tag{3}$$

in the sense of $\mathscr{D}'(\mathbb{R})$. What is the general solution $u \in \mathscr{D}'(\mathbb{R})$ to (3)?