

Problem Sheet 3

Problem 1. Find the general solutions to the ODEs

(i)

$$y'' + 2y' + y = 1$$

(ii)

$$y'' + 2y' + y = H$$

(iii)

$$y'' + 2y' + y = \delta_0$$

in $\mathcal{D}'(\mathbb{R})$, where H is Heaviside's function and δ_0 is Dirac's delta-function at 0. What are the classical solutions to (i) and (ii)?

Problem 2. The principal logarithm is defined on the cut plane $\mathbb{C} \setminus (-\infty, 0]$ as

$$\text{Log} z := \log |z| + i\text{Arg}(z), \quad \text{Arg}(z) \in (-\pi, \pi).$$

Define $\text{Log}(x + i0)$ and $\text{Log}(x - i0)$ for each $\varphi \in \mathcal{D}(\mathbb{R})$ by the rules

$$\langle \text{Log}(x \pm i0), \varphi \rangle := \lim_{\varepsilon \searrow 0} \int_{-\infty}^{\infty} \text{Log}(x \pm i\varepsilon) \varphi(x) dx.$$

(a) Show that $\text{Log}(x \pm i0)$ hereby are distributions on $\mathbb{R}$.

Now let $k \in \mathbb{N}$ and define the distributions $(x + i0)^{-k}$ and $(x - i0)^{-k}$ as

$$(x \pm i0)^{-k} := \frac{(-1)^{k-1}}{(k-1)!} \frac{d^k}{dx^k} \text{Log}(x \pm i0) \quad \text{in } \mathcal{D}'(\mathbb{R}).$$

(b) Show that for each $\varphi \in \mathcal{D}(\mathbb{R})$ with $\varphi^{(j)}(0) = 0$ for $j \in \{0, \dots, k\}$ we have

$$\langle (x \pm i0)^{-k}, \varphi \rangle = \int_{-\infty}^{\infty} \frac{\varphi(x)}{x^k} dx.$$

(c) Prove that $\text{Log}(x + i0) - \text{Log}(x - i0) = 2\pi i \tilde{H}$, where H is the Heaviside function. Deduce the *Plemelj-Sokhotsky jump relations*:

$$(x + i0)^{-k} - (x - i0)^{-k} = 2\pi i \frac{(-1)^k}{(k-1)!} \delta_0^{(k-1)},$$

where δ_0 is Dirac's delta-function on $\mathbb{R}$ concentrated at 0.

(d) Show that

$$x(x \pm i0)^{-1} = 1 \quad \text{in } \mathcal{D}'(\mathbb{R}).$$

Deduce that

$$(x + i0)^{-1}(x\delta_0) = 0 \neq \delta_0 = \left((x + i0)^{-1}x \right) \delta_0.$$

Next, show, for instance by using the differential operator $x \frac{d}{dx}$ on the case $k = 1$ iteratively, that

$$x^k(x \pm i0)^{-k} = 1 \quad \text{in } \mathcal{D}'(\mathbb{R})$$

holds for each $k \in \mathbb{N}$.

Problem 3. Let $g \in L^1_{\text{loc}}(\mathbb{R})$ and assume that g is T periodic for some $T > 0$: $g(x + T) = g(x)$ holds for almost all $x \in \mathbb{R}$. Define for each $j \in \mathbb{N}$ the function

$$g_j(x) = g(jx), \quad x \in (0, 1).$$

Prove that

$$g_j \rightarrow \frac{1}{T} \int_0^T g \, dx \quad \text{in } \mathcal{D}'(0, 1) \quad \text{as } j \rightarrow \infty.$$

Problem 4. Let $\theta \in \mathcal{D}'(\mathbb{R})$.

(i) Explain how the convolution $\theta * u$ is defined for a general distribution $u \in \mathcal{D}'(\mathbb{R})$.

(ii) Prove that $\theta * u \in C^\infty(\mathbb{R})$ when $u \in \mathcal{D}'(\mathbb{R})$.

(iii) Let $(\rho_\varepsilon)_{\varepsilon > 0}$ be the standard mollifier on $\mathbb{R}$. Show that for a general distribution $u \in \mathcal{D}'(\mathbb{R})$ we have that

$$\rho_\varepsilon * u \rightarrow u \quad \text{in } \mathcal{D}'(\mathbb{R}) \quad \text{as } \varepsilon \searrow 0.$$

(iv) Show that for each $u \in \mathcal{D}'(\mathbb{R})$ we can find a sequence (u_j) in $\mathcal{D}(\mathbb{R})$ such that

$$u_j \rightarrow u \quad \text{in } \mathcal{D}'(\mathbb{R}) \quad \text{as } j \rightarrow \infty.$$

Problem 5. Let

$$p(\partial) = \sum_{|\alpha| \leq k} c_\alpha \partial^\alpha \quad (k \in \mathbb{N} \text{ and } c_\alpha \in \mathbb{C})$$

be a partial differential operator on $\mathbb{R}^n$ in the usual multi-index notation. For an open subset Ω of $\mathbb{R}^n$ and $u \in \mathcal{D}'(\Omega)$ show that the supports always obey the rule:

$$\text{supp}(p(\partial)u) \subseteq \text{supp}(u).$$

Give an example of a distribution $v \in \mathcal{D}'(\mathbb{R})$ such that the distributional derivative $v' \neq 0$ has compact support, but v itself hasn't.

Next, show that also the singular supports satisfy the rule

$$\text{sing. supp}(p(D)u) \subseteq \text{sing. supp}(u)$$

and give an example of a distribution $u \in \mathcal{D}'(\mathbb{R}^2)$ and a partial differential operator $p(\partial)$ so

$$\text{sing. supp}(u) = \mathbb{R}^2 \text{ and } \text{sing. supp}(p(\partial)u) = \emptyset.$$

Problem 6. (Optional)

Let $F: \mathbb{C} \rightarrow \mathbb{C}$ be an entire function that is not identically zero. Explain why the formula $f = \log |F|$ defines a distribution on $\mathbb{C}$.

Prove that its distributional Laplacian equals

$$\Delta f = \sum_{j \in J} 2\pi m_j \delta_{z_j}$$

where $\{z_j : j \in J\}$ are the distinct zeros for F and $\{m_j : j \in J\}$ their multiplicities.

[Hint: Use the Cauchy-Riemann operators to calculate the Laplacian.]