

Problem Sheet 4

Problem 1. *This question provides a condition ensuring that the usual partial derivatives coincide with the distributional partial derivatives. Prove Lemma 5.21 from the lecture notes: If the dimension $n \geq 2$ and $f \in C^1(\mathbb{R}^n \setminus \{0\}) \cap L^1_{\text{loc}}(\mathbb{R}^n)$ has usual partial derivatives $\partial_j f \in L^1_{\text{loc}}(\mathbb{R}^n)$ for each direction $1 \leq j \leq n$, then also*

$$\int_{\mathbb{R}^n} \partial_j f \varphi \, dx = - \int_{\mathbb{R}^n} f \partial_j \varphi \, dx$$

holds for all $\varphi \in \mathcal{D}(\mathbb{R}^n)$. Give an example to show that it can fail for dimension $n = 1$. Show that for dimension $n = 1$ we instead have the following: If $f \in C^1(\mathbb{R} \setminus \{0\}) \cap C(\mathbb{R})$ and the usual derivative $f' \in L^1_{\text{loc}}(\mathbb{R})$, then

$$\int_{\mathbb{R}} f' \varphi \, dx = - \int_{\mathbb{R}} f \varphi' \, dx$$

holds for all $\varphi \in \mathcal{D}(\mathbb{R})$.

Problem 2. *Boundary values in the sense of distributions for holomorphic functions.*

(a) Prove that for each $n \in \mathbb{N}$,

$$(x + i\varepsilon)^{-n} \rightarrow (x + i0)^{-n} \quad \text{in } \mathcal{D}'(\mathbb{R}) \text{ as } \varepsilon \searrow 0,$$

where the distribution $(x + i0)^{-n}$ was defined in Problem 2 on Sheet 3.

A holomorphic function $f: H \rightarrow \mathbb{C}$ on the upper half-plane $H = \{z \in \mathbb{C} : \text{Im}(z) > 0\}$ is said to be of slow growth if for each $R > 0$ there exist $m = m_R \in \mathbb{N}_0$ and $c = c_R \geq 0$ so

$$|f(z)| \leq \frac{c}{\text{Im}(z)^m}$$

holds for all $z \in H$ with $|\text{Re}(z)| \leq R$ and $\text{Im}(z) < 2$.

(b) Prove that if $f: H \rightarrow \mathbb{C}$ is holomorphic of slow growth, then it has a boundary value in the sense of distributions:

$$\langle f(x + i0), \varphi \rangle := \lim_{\varepsilon \searrow 0} \int_{\mathbb{R}} f(x + i\varepsilon) \varphi(x) \, dx$$

exists for all $\varphi \in \mathcal{D}(\mathbb{R})$ and defines a distribution. [Hint: Assume first that $m = 0$ above and let $F: H \rightarrow \mathbb{C}$ be the holomorphic primitive with $F(i) = 0$. Explain why F has a continuous extension to the closed upper half-plane $\overline{H}$ and use this to conclude the proof in this special case. Then use induction on m .]

Problem 3. Distributions defined by finite parts.

Recall from Sheet 2 that the distributional derivative of $\log|x|$ is the distribution $\text{pv}(\frac{1}{x})$ defined by the principal value integral

$$\langle \text{pv}(\frac{1}{x}), \varphi \rangle := \lim_{\varepsilon \searrow 0} \left(\int_{-\infty}^{-\varepsilon} + \int_{\varepsilon}^{\infty} \right) \frac{\varphi(x)}{x} dx, \quad \varphi \in \mathcal{D}(\mathbb{R}).$$

In order to represent the higher order derivatives one can use finite parts: Let $n \in \mathbb{N}$ with $n > 1$. We then define $\text{fp}(\frac{1}{x^n})$ for each $\varphi \in \mathcal{D}(\mathbb{R})$ by the *finite part integral*

$$\langle \text{fp}(\frac{1}{x^n}), \varphi \rangle := \int_{-\infty}^{\infty} \frac{\varphi(x) - \sum_{j=0}^{n-2} \frac{\varphi^{(j)}(0)}{j!} x^j - \frac{\varphi^{(n-1)}(0)}{(n-1)!} x^{n-1} \mathbf{1}_{(-1,1)}(x)}{x^n} dx.$$

(a) Check that hereby $\text{fp}(\frac{1}{x^n})$ is a well-defined distribution on $\mathbb{R}$. Show that

$$\frac{d}{dx} \text{pv}(\frac{1}{x}) = -\text{fp}(\frac{1}{x^2}) \quad \text{and} \quad \frac{d}{dx} \text{fp}(\frac{1}{x^n}) = -n \text{fp}(\frac{1}{x^{n+1}})$$

for all $n > 1$. Is $\text{fp}(\frac{1}{x^n})$ homogeneous? (See Problem 4 on Sheet 2 for the definition of homogeneity.)

(b) Show that for $n > 1$ we have $x^n \text{fp}(\frac{1}{x^n}) = 1$ and find the general solution to the equation $x^n u = 1$ in $\mathcal{D}'(\mathbb{R})$. What is the general solution to the equation $(x-a)^n v = 1$ in $\mathcal{D}'(\mathbb{R})$ when $a \in \mathbb{R} \setminus \{0\}$?

(c) *Optional.* Let $p(x) \in \mathbb{C}[x] \setminus \{0\}$ be a nontrivial polynomial. Describe the general solution $w \in \mathcal{D}'(\mathbb{R})$ to the equation

$$p(x)w = 1 \text{ in } \mathcal{D}'(\mathbb{R}).$$

Problem 4. A function $f: \mathbb{R} \rightarrow \mathbb{R}$ is *convex* if for all $x_0, x_1 \in \mathbb{R}$ and $\lambda \in (0, 1)$ we have

$$f(\lambda x_1 + (1-\lambda)x_0) \leq \lambda f(x_1) + (1-\lambda)f(x_0). \quad (1)$$

A function $a: \mathbb{R} \rightarrow \mathbb{R}$ satisfying (1) with equality everywhere is called an *affine function*.

(a) Show that an affine function must have the form $a(x) = a_1 x + a_0$ for some constants $a_0, a_1 \in \mathbb{R}$. Show also that a function $f: \mathbb{R} \rightarrow \mathbb{R}$ is convex if and only if it for each compact interval $[\alpha, \beta] \subset \mathbb{R}$ has the property:

$$\text{when } a \text{ is affine and } f(x) \leq a(x) \text{ for } x \in \{\alpha, \beta\}, \text{ then } f \leq a \text{ on } [\alpha, \beta]$$

(b) Show that a convex function $f: \mathbb{R} \rightarrow \mathbb{R}$ satisfies the *3 slope inequality*:

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} \leq \frac{f(x_3) - f(x_1)}{x_3 - x_1} \leq \frac{f(x_3) - f(x_2)}{x_3 - x_2}$$

holds for all triples $x_1 < x_2 < x_3$. Deduce that a convex function must be continuous and that it is differentiable except for in at most countably many points.

Optional: Show that a convex function must be locally Lipschitz continuous: for each $r > 0$ there exists $L = L_r \geq 0$ so $|f(x) - f(y)| \leq L|x - y|$ holds for all $x, y \in [-r, r]$.

(c) Assume that $f: \mathbb{R} \rightarrow \mathbb{R}$ is twice differentiable. Show that f is convex if and only if

$$f''(x) \geq 0$$

holds for all $x \in \mathbb{R}$.

(d) Let $u \in \mathcal{D}'(\mathbb{R})$ and assume that $u'' \geq 0$ in $\mathcal{D}'(\mathbb{R})$. Show that u is represented by a convex function.