1. (a) [9 marks] State the definitions of *test function* $\varphi \in \mathscr{D}(\mathbb{R}^n)$ and *distribution* $u \in \mathscr{D}'(\mathbb{R}^n)$. We now assume that $u \in \mathscr{D}'(\mathbb{R}^n)$ and $j \in \{1, \ldots, n\}$. Explain how one should define its *distributional partial derivative* $\partial_j u$ and show that it reduces to the usual concept for C^1 functions.

   Suppose that $\theta \in \mathscr{D}(\mathbb{R}^n)$ and let $\left(\rho_\varepsilon\right)_{\varepsilon>0}$ be the standard mollifier on $\mathbb{R}^n$.

   (i) Define $u * \theta$ and explain how to prove that $u * \theta \in \mathrm{C}^\infty(\mathbb{R}^n)$, $(u * \theta)(x) = \langle u, \theta(x - \cdot)\rangle$ and $\partial_j(u * \theta) = (\partial_j u) * \theta = u * (\partial_j \theta)$.

   (ii) Explain why $\rho_\varepsilon * u \to u$ in $\mathscr{D}'(\mathbb{R}^n)$ as $\varepsilon \searrow 0$.

   [*Detailed arguments are not required in (i), (ii) and it suffices to merely list the main steps of the proofs.*]

   State and prove the *constancy theorem* for distributions on $\mathbb{R}^n$.

   (b) [10 marks] Define the product au for $a \in \mathrm{C}^\infty(\mathbb{R})$ and $u \in \mathscr{D}'(\mathbb{R})$. State the *Leibniz rule* for differentiation of au. [*No proof is required.*]

   (i) Solve the equation
   $$u' + 2xu = \delta_0$$
   for $u \in \mathscr{D}'(\mathbb{R})$, where δ_0 is Dirac's delta function at 0.

   (ii) Let $f \in \mathscr{D}'(\mathbb{R})$. Solve the equation
   $$v' + 2xv = f$$
   for $v \in \mathscr{D}'(\mathbb{R})$ [*The fundamental theorem of calculus for distributions on* $\mathbb{R}$ *can be used without proof provided you state it clearly.*]

   (iii) What is the general solution to
   $$w' - 2xw = 0$$
   in the space of *compactly supported distributions* $w \in \mathscr{E}'(\mathbb{R})$?

   (c) [6 marks] Prove that $u \in \mathrm{W}^{1,\infty}(0,1)$ if and only if $u \in \mathrm{L}^\infty(0,1)$ admits a Lipschitz continuous representative. [*You may use without proof that any sequence* (f_j) *that is bounded in* $\mathrm{L}^\infty(0,1)$ *admits a subsequence* (f_{j_k}) *so that for some* $f \in \mathrm{L}^\infty(0,1)$ *we have* $f_{j_k} \to f$ *in* $\mathscr{D}'(0,1)$.]

2. (a) [9 marks] For each $r > 0$ we define the r-dilation of a test function $\varphi \in \mathscr{D}(\mathbb{R}^n)$ by the rule

$$(d_r\varphi)(x) = \varphi(rx), \quad x \in \mathbb{R}^n.$$

(i) Extend the r-dilation to distributions $u \in \mathscr{D}'(\mathbb{R}^n)$ in a consistent manner.

(ii) Let $\alpha \in (-n, \infty)$ and $u_\alpha(x) = |x|^\alpha$ for $x \in \mathbb{R}^n \setminus \{0\}$. Show that u_α is a *regular distribution* on $\mathbb{R}^n$. Prove that $d_r u_\alpha = r^\alpha u_\alpha$ for all $r > 0$. We express this by saying that u_α is *homogeneous of degree* α.

(iii) Show that the Dirac delta function δ_0 concentrated at the origin $0 \in \mathbb{R}^n$ is homogeneous of degree $-n$.

(b) [8 marks] Define for each $\varphi \in \mathscr{D}(\mathbb{R})$,

$$\left\langle \mathrm{pv}\Big(\frac{1}{x}\Big), \varphi \right\rangle = \lim_{a \searrow 0} \left(\int_{-\infty}^{-a} + \int_{a}^{\infty} \right) \frac{\varphi(x)}{x}\, \mathrm{d}x.$$

Show that hereby $\mathrm{pv}\big(\frac{1}{x}\big) \in \mathscr{D}'(\mathbb{R})$ and that it is homogeneous of degree -1 (as defined in (a) above). Show that

$$\frac{\mathrm{d}}{\mathrm{d}x} \log|x| = \mathrm{pv}\Big(\frac{1}{x}\Big).$$

What is the order of $\mathrm{pv}\big(\frac{1}{x}\big)$?

(c) [8 marks] Show that $u = \frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{pv}\big(\frac{1}{x}\big)\right)$ solves the equation

$$x^2 u = -1 \tag{1}$$

in the sense of $\mathscr{D}'(\mathbb{R})$. What is the general solution $u \in \mathscr{D}'(\mathbb{R})$ to (1)? [*Results from the course can be used without proof provided they are clearly stated.*]

3. (a) [10 marks] Let $E = \frac{1}{2\pi} \log|z|$, where $z = x+iy \in \mathbb{C} \approx \mathbb{R}^2$. Explain why E is a distribution on $\mathbb{R}^2$ and show that $\Delta E = \delta_0$ in $\mathscr{D}'(\mathbb{R}^2)$, where δ_0 is Dirac's delta function at 0.

Using for instance that $\Delta = 4\partial^2/\partial z \partial \bar{z}$ deduce that

$$\frac{\partial}{\partial \bar{z}} \left(\frac{1}{\pi z} \right) = \delta_0 \text{ in } \mathscr{D}'(\mathbb{R}^2).$$

Let $f \in \mathscr{E}'(\mathbb{R}^2)$. Find the general solution to the PDE

$$\frac{\partial u}{\partial \bar{z}} = f \text{ in } \mathscr{D}'(\mathbb{R}^2).$$

Explain why all solutions u will be C^∞ on an open subset ω of $\mathbb{R}^2$ if the right-hand side f is C^∞ on ω. [*Results from the course can be used without proof provided they are clearly stated.*]

(b) [4 marks] Let $G\colon \mathbb{C} \to \mathbb{C}$ be an entire function satisfying $G(z) \neq 0$ for all $z \in \mathbb{C}$. Explain why $g = \log|G| \in \mathscr{D}'(\mathbb{R}^2)$.

Prove that $\Delta g = 0$ holds in both the usual and the distributional sense.

(c) [5 marks] Let $F\colon \mathbb{C} \to \mathbb{C}$ be an entire function such that $F(z) = 0$ if and only if $z = 0$. Put $f = \log|F|$. Explain why $f \in \mathscr{D}'(\mathbb{R}^2)$ and compute Δf in $\mathscr{D}'(\mathbb{R}^2)$.

(d) [6 marks] Let P and N be finite subsets of $\mathbb{C}$ and $H\colon \mathbb{C}\setminus P \to \mathbb{C}$ be a holomorphic function with poles in P and zeros precisely in N. Explain why $h = \log|H|$ defines a distribution on $\mathbb{R}^2$ and compute Δh in $\mathscr{D}'(\mathbb{R}^2)$. [*Results from the course can be used without proof provided they are clearly stated.*]