Distribution Theory and Analysis of PDEs **MT19**

Problem Sheet 1

Problem 1. Define $\phi\colon \mathbb{R} \to \mathbb{R}$ by

$$\phi(x) = \begin{cases} e^{-\frac{1}{x}} & \text{if } x > 0 \\ 0 & \text{if } x \leq 0. \end{cases}$$

Show that ϕ is C^∞, and deduce that

$$\psi(x) = \phi\big(2(1-x)\big)\phi\big(2(1+x)\big)$$

belongs to $\mathscr{D}(\mathbb{R})$. Does the restriction to $(-1,1)$, $\psi|_{(-1,1)}$, belong to $\mathscr{D}(-1,1)$?

Calculate the Taylor series for ϕ about 0 (note: not for ψ). Does the series converge, and if so, then what is its sum?

Problem 2. In this question all functions and distributions are real-valued.
(a) Let K be a compact proper subset of the open interval (a,b). Show carefully that there exists $\rho \in \mathscr{D}(a,b)$ such that $0 \leq \rho \leq 1$ and $\rho = 1$ on K.
(b) Give an example of $\varphi, \psi \in \mathscr{D}(\mathbb{R})$ such that $\max(\varphi,\psi)$, $\min(\varphi,\psi)$ are *not* smooth compactly supported test functions. Here we define $\max(\varphi,\psi)(x) = \max\{\varphi(x),\psi(x)\}$ for each x and similarly for $\min(\varphi,\psi)$.

Next, let $u \in \mathscr{D}(a,b)$. Show that there exist u_1, $u_2 \in \mathscr{D}(a,b)$ with $u_1 \geq 0$, $u_2 \geq 0$ and $u = u_1 - u_2$.
(c) Generalize the last statement to n dimensions as follows. Let Ω be a nonempty open subset of $\mathbb{R}^n$ and $u \in \mathscr{D}(\Omega)$. Show that there exist u_1, $u_2 \in \mathscr{D}(\Omega)$ with $u_1 \geq 0$ and $u_2 \geq 0$ such that $u = u_1 - u_2$.
(*Hint: You may for instance note that* $4u = (u+1)^2 - (u-1)^2$ *and if* v *is a cut-off function between the support of* u *and the boundary of* Ω*, then* $vu = u$.)

Problem 3. Let Ω be a nonempty and open subset of $\mathbb{R}^n$, $1 \leq p < \infty$ and $f \in L^p(\Omega)$. Show that for each $\varepsilon > 0$ there exists $g \in \mathscr{D}(\Omega)$ such that $\|f-g\|_p < \varepsilon$.
(*Hint: One approach is to do it in two steps. First choose an appropriate open subset* $O \subset \Omega$ *so that* $h = f\mathbf{1}_O$ *is a good* L^p *approximation of* f*. Then use a result from lectures.*)

Problem 4. Let p, $q \in [1,\infty]$ with $\frac{1}{p} + \frac{1}{q} = 1$. Show that if $f \in L^p(\mathbb{R})$, $g \in L^q(\mathbb{R})$, then $f * g \in C(\mathbb{R})$. Next, show that if $p \in (1,\infty)$, then $f * g \in C_0(\mathbb{R})$, that is, $f * g$ is continuous and $(f*g)(x) \to 0$ as $|x| \to \infty$. What happens when $p = 1$ and $q = \infty$?

Note it has been changed a bit!

1. ϕ clearly C^∞ on $(-\infty,0)\cup(0,\infty)$ with $\phi^{(k)}(x)=0$ for $x<0$. For $x>0$,

$\phi'(x) = \frac{1}{x^2}e^{-\frac{1}{x}}$, $\phi''(x) = \frac{-2x+1}{x^4}e^{-\frac{1}{x}}$ and

then claim: $\phi^{(k)}(x) = \frac{p_k(x)}{x^{2k}}e^{-\frac{1}{x}}$, $x>0$,

where p_k real polynomial of degree at most $k-1$. By induction: ok for $k=1,2$, if true for k, then $\phi^{(k+1)}(x) = \frac{p_k'(x)x^2 - 2kp_k(x)x + p_k(x)}{x^{2(k+1)}}$

$\times e^{-\frac{1}{x}}$ and $p_{k+1}(x) := p_k'(x)x^2 - 2kp_k(x)x + p_k(x)$ is a real polynomial of degree at most $\deg(p_k)+1 \leq k$.

Claim ϕ infinitely often diff. at $x=0$ with $\phi^{(k)}(0)=0$ for all k.

Induction on k: $k=1$

For $x>0$ we have by Mean Value Theorem

$$\frac{\phi(x)-\phi(0)}{x} = \phi'(\theta) \quad \text{for some } \theta\in(0,x).$$

Now $\phi'(\theta) = \frac{1}{\theta^2}e^{-\frac{1}{\theta}}$ and $e^t > \frac{t^3}{6}$ for $t>0$

so $e^{-\frac{1}{\theta}} < \frac{6}{\left(\frac{1}{\theta}\right)^3} = 6\theta^3$ and

thus $\phi'(\theta) < 6\theta < x \longrightarrow 0$ as $x \to 0+$.

It follows that $\lim_{x \to 0} \frac{\phi(x) - \phi(0)}{x} = 0$ so ϕ

is diff. at 0 with $\phi'(0) = 0$.

Assume ϕ is k times diff. at 0 with $\phi^{(k)}(0) = 0$. Then

$$\phi^{(k)}(x) = \begin{cases} \frac{P_k(x)}{x^{2k}} e^{-\frac{1}{x}}, & x > 0 \\ 0, & x \leq 0. \end{cases}$$

For $x > 0$ we get as above:

$\frac{1}{x}\left(\phi^{(k)}(x) - \phi^{(k)}(0)\right) = \phi^{(k+1)}(\theta)$ for a $\theta \in (0, x)$.

Note $e^t > \frac{t^{2k+3}}{(2k+3)!}$ for $t > 0$ so that

$$\phi^{(k+1)}(\theta) = \frac{P_{k+1}(\theta)}{\theta^{2k+2}} e^{-\frac{1}{\theta}} < \frac{P_{k+1}(\theta)}{\theta^{2k+2}} \frac{(2k+3)!}{\left(\frac{1}{\theta}\right)^{2k+3}} =$$

$(2k+3)!\, P_{k+1}(\theta)\, \theta \longrightarrow 0$ as $x \to 0+$.

It follows that $\lim_{x \to 0} \frac{\phi^{(k)}(x) - \phi^{(k)}(0)}{x} = 0$

so $\phi^{(k)}$ is diff. at 0 with $\phi^{(k+1)}(0) = 0$. $\square$

By Leibniz ψ is C^∞, and $\psi(x) \neq 0$ iff

$2(1-x)>0$ and $2(1+x)>0$, that is, iff $-1<x<1$. Thus $\mathrm{spt}(\psi)=[-1,1]$, and therefore $\psi\in\mathcal{D}(\mathbb{R})$.

Functions in $\mathcal{D}(-1,1)$ must have compact support in $(-1,1)$, so $\psi|_{(-1,1)}\notin\mathcal{D}(-1,1)$.

Taylor series for ψ about 0 is $\sum_{n=0}^{\infty} 0x^n=0$.

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(2) (a) Let $\psi\in\mathcal{D}(\mathbb{R})$ be as in (1) and put $\phi=\dfrac{\psi}{\int_{\mathbb{R}}\psi\,dx}$. Then $\phi\in\mathcal{D}(\mathbb{R})$, $\phi\geq 0$, $\mathrm{spt}(\phi)=[-1,1]$ and $\int\phi\,dx=1$. Put $\phi_\varepsilon(x)=\varepsilon^{-1}\phi(\varepsilon^{-1}x)$ so that $(\phi_\varepsilon)_{\varepsilon>0}$ is standard mollifier. Because $K\subset(a,b)$ is compact $d=\mathrm{dist}(K,\{a,b\})>0$ and we can take $\rho(x)=(\phi_\varepsilon * 1_{B_{\frac{d}{2}}(K)})(x)$ for $\varepsilon<\frac{d}{4}$:

By result from lectures $\rho\in C^\infty(\mathbb{R})$.

If $x\notin\overline{B_{\frac{3d}{4}}(K)}$, so $\mathrm{dist}(x,\overline{B_{\frac{d}{2}}(K)})>\frac{d}{4}$, then $\rho(x) = \int_{\overline{B_{\frac{d}{2}}(K)}} \phi_\varepsilon(x-y)\,dy = 0$ since

$\phi_\varepsilon(x-y) > 0$ iff $|x-y| < \varepsilon$, but when

$y \in \overline{B_{\frac{d}{2}}(K)}$ we have

$$\frac{d}{4} < \operatorname{dist}\left(x, \overline{B_{\frac{d}{2}}(K)}\right) = \operatorname{dist}\left(x, \overline{B_{\frac{d}{2}}(K)}\right) - \operatorname{dist}\left(y, \overline{B_{\frac{d}{2}}(K)}\right)$$
$$\leq |x-y|.$$

$\therefore$ $\operatorname{spt}(\rho) \subseteq \overline{B_{\frac{3d}{4}}(K)} \subset (a,b)$.

If $x \in K$, then $B_\varepsilon(x) \subset \overline{B_{\frac{d}{2}}(K)}$ and so

$$\rho(x) = \int_{\overline{B_{\frac{d}{2}}(K)}} \phi_\varepsilon(x-y)\,dy = \int_{\mathbb{R}} \phi_\varepsilon(x-y)\,dy = 1.$$

Finally, $0 \leq \rho(x) \leq \int_{\mathbb{R}} \phi_\varepsilon = 1$ for all x.

(b) Let $\psi(x) = \begin{cases} e^{\frac{1}{1-x^2}}, & |x| < 1 \\ 0, & |x| \geq 1 \end{cases}$ from (a)

and $\varphi(x) = \psi(x-1)$. Then $\varphi(x) = \psi(x-1)$ iff

$x = \frac{1}{2}$ and

$$\max(\varphi, \psi) = \begin{cases} \varphi & x > \frac{1}{2} \\ \psi & x \leq \frac{1}{2} \end{cases}, \quad \min(\varphi, \psi) = \begin{cases} \psi & x > \frac{1}{2} \\ \varphi & x \leq \frac{1}{2} \end{cases}$$

Since $\varphi'(\frac{1}{2}) = \psi'(-\frac{1}{2}) = -\psi'(\frac{1}{2})$ and

$\psi'(\frac{1}{2}) = \frac{16}{9} e^{\frac{4}{3}} \neq 0$ so $\max(\varphi, \psi), \min(\varphi, \psi)$ are not diff. at $x=\frac{1}{2}$, hence not in $\mathcal{D}(\mathbb{R})$. 5/18

(c) 1-dimensional case follows also from:

Take $K = \mathrm{spt}(u)$ so that $K \subset \Omega$ compact.

Let $\rho \in \mathcal{D}(\Omega)$, $0 \leq \rho \leq 1$ and $\rho = 1$ on K be the cut-off function constructed in Theorem 2.4 from Lectures. Define

$$u_1 = \rho\frac{(u+1)^2}{4}, \quad u_2 = \rho\frac{(u-1)^2}{4}.$$

Then $u_1, u_2 \in \mathcal{D}(\Omega)$, $u_1, u_2 \geq 0$ and $u_1 - u_2 = u$.

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(3) Put $\Omega_k = \{x \in \Omega \cap B_k(0) : \mathrm{dist}(x, \partial\Omega) > \frac{1}{k}\}$ for $k \in \mathbb{N}$. Then by MCT

$$\int_\Omega |f - f\mathbb{1}_{\Omega_k}|^p \, dx = \int_{\Omega \setminus \Omega_k} |f|^p \, dx \xrightarrow[k\to\infty]{} 0.$$

Take $k \in \mathbb{N}$ so $\left(\int_{\Omega\setminus\Omega_k} |f|^p \, dx\right)^{\frac{1}{p}} < \frac{\varepsilon}{2}$,

put $O = \Omega_k$ and $h = f\mathbb{1}_O$. Then $h \in L^p(\mathbb{R}^n)$

and by Proposition 2.2 from lectures

$\rho_\delta * h \in C^\infty(\Omega)$, $\|\rho_\delta * h - h\|_p \to 0$ as $\delta \to 0+$,

where $(\rho_\delta)_{\delta>0}$ is the standard mollifier.

Since $h = 0$ a.e. on $\Omega \setminus O \subset \{x \in \Omega : \mathrm{dist}(x, \partial\Omega) \leq \frac{1}{k}\}$

we have $\rho_\delta * h(x) = 0$ when $\mathrm{dist}(x, \partial\Omega) < \frac{1}{k} - 2\delta$

Take $\delta \in (0, \frac{1}{4k})$ so small that

$\|\rho_\delta * h - h\|_p < \frac{\varepsilon}{2}$.

Hereby $\mathrm{spt}(\rho_\delta * h) \subseteq \{x \in \Omega \cap \overline{B_{k+2\delta}(0)} : \mathrm{dist}(x, \partial\Omega) \geq \frac{1}{2k}\}$

$\subset \Omega$ so that $\rho_\delta * h$ has compact support in Ω. Thus $g = \rho_\delta * h \in \mathcal{D}(\Omega)$ and from

Minkowski:

$\|f - g\|_p \leq \|f - h\|_p + \|h - g\|_p < \varepsilon$. $\square$

(4) Let $f \in L^p(\mathbb{R})$, $g \in L^q(\mathbb{R})$, where $\frac{1}{p} + \frac{1}{q} = 1$. $\frac{7}{18}$

For $x \in \mathbb{R}$ fixed, the function

$$y \mapsto |f(x-y)g(y)|$$

is measurable (pick representatives for f and g) and integrable by Hölder

$$\int_{\mathbb{R}} |f(x-y)g(y)|\,dy \leq \|f(x-\cdot)\|_p \|g\|_q = \|f\|_p \|g\|_q < \infty.$$

Consequently, $(f*g)(x)$ is well-defined (and does not depend on representatives used for calculating the integral). Next, for $x, x_0 \in \mathbb{R}$ we then find, again using Hölder,

$$|(f*g)(x) - (f*g)(x_0)| \leq \int_{\mathbb{R}} |f(x-y) - f(x_0-y)||g(y)|dy$$

$$\leq \|\tau_{-x}\tilde{f} - \tau_{-x_0}\tilde{f}\|_p \|g\|_q.$$

$\boxed{\text{Notation: } \tilde{f}(x) := f(-x) \text{ and } \tau_h f(x) := f(x+h)}$

If $p < \infty$ we have from Lemma 2.9 that $\|\tau_{-x}\tilde{f} - \tau_{-x_0}\tilde{f}\|_p \to 0$ as $x \to x_0$ (more precisely it clearly holds when $f \in C_c(\mathbb{R})$ and according to Lemma 2.9 $C_c(\mathbb{R})$ is dense in $L^p(\mathbb{R})$ when $p<\infty$) When $p=\infty$ we use instead that

$$|(f*g)(x)-(f*g)(x_0)| \leq \int_{\mathbb{R}} |f(y)||g(x-y)-g(x_0-y)|dy$$

$$\leq \|f\|_\infty \|\tau_x \tilde{g} - \tau_{x_0}\tilde{g}\|_1$$

$\rightarrow 0$ as $x \rightarrow x_0$. Thus $f*g$ is continuous.

Assume $1<p<\infty$ (so also $1<q<\infty$) and let $\varepsilon>0$. Use Lemma 2.9 to find $h \in C_c(\mathbb{R})$ with $\|f-h\|_p < \varepsilon$. Then $|(f*g)(x)-(h*g)(x)| \leq \int_{\mathbb{R}} |f(y)-h(y)||g(x-y)|dy \overset{\text{Hölder}}{\leq} \|f-h\|_p \|\tau_x \tilde{g}\|_q \leq \varepsilon \|g\|_q$. Take $a>0$ so $\mathrm{supp}(h) \subseteq [-a,a]$ and note $|(h*g)(x)| \leq \int_{\mathbb{R}} |h(y)||g(x-y)|dy \leq \|h\|_\infty \int_{x-a}^{x+a} |g(y)|dy \longrightarrow 0$ as $|x| \rightarrow \infty$ (indeed, if $\int_{x-a}^{x+a}|g| \not\rightarrow 0$ then $\exists \delta>0$ and $|x_j| \rightarrow \infty$ so $\int_{x_j-a}^{x_j+a}|g| \geq \delta$ $\forall j$. But for a subseq. (x_{j_k}) the intervals $(x_{j_k}-a, x_{j_k}+a)$ are disjoint so $\sum_k \int_{x_{j_k}-a}^{x_{j_k}+a}|g|^q \leq \int_{\mathbb{R}}|g|^q < \infty$ and $\frac{1}{2a}\int_{x_{j_k}-a}^{x_{j_k}+a}|g| \leq (\frac{1}{2a}\int_{x_{j_k}-a}^{x_{j_k}+a}|g|^q)^{\frac{1}{q}}$ $\lightning$). This is <u>false</u> for $f \in L^1(\mathbb{R})$, $g \in L^\infty(\mathbb{R})$: take $f \in L^1(\mathbb{R})$, $f \not\equiv 0$, $g = \mathbb{1}_{\mathbb{R}} \in L^\infty(\mathbb{R})$.

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(5) u_1 isn't a distribution as it's not of locally finite order:

Let $\rho \in \mathcal{D}(\mathbb{R})$ be a cut-off function satisfying $0 \leq \rho \leq 1$, $\rho = 1$ on $(-\frac{1}{2}, \frac{1}{2})$ and $\operatorname{spt}(\rho) \subseteq (-1,1)$. Then $\rho(0) = 1$ but $\rho^{(k)}(0) = 0$ $\forall k \geq 1$.

If u_1 is a distribution, then there exist $c>0$, $m\in\mathbb{N}_0$ so

$$|\langle u_1,\varphi\rangle| = |\sum_{j=0}^{\infty} 2^{-j}\varphi^{(j)}(0)| \leq c\sum_{s=0}^{m} \sup_{[-1,1]} |\varphi^{(s)}|$$

For $n\in\mathbb{N}$ put $\varphi(x) = x^n\rho(x)$, $x\in\mathbb{R}$. Then $\varphi\in\mathcal{D}(-1,1)$ and by Leibniz:

$$\varphi^{(k)}(x) = \sum_{j=0}^{k}\binom{k}{j}\frac{\partial^j}{\partial x^j}(x^n)\,\rho^{(k-j)}(x).$$

In particular $\varphi^{(k)}(0) = \begin{cases} n! & \text{if } k=n \\ 0 & \text{if } k\neq n\end{cases}$

and $$|\varphi^{(k)}(x)| \leq \sum_{j=0}^{k}\binom{k}{j} n(n-1)\cdots(n-j+1)|\rho^{(k-j)}(x)|$$

$$\leq 2^k n(n-1)\cdots(n-k+1)\max_{0\leq j\leq k}\sup|\rho^{(j)}(x)|,$$

hence

$$\text{RHS} \leq c\sum_{s=0}^{m}\sup_{[-1,1]}|\varphi^{(s)}| \leq c\sum_{s=0}^{m} 2^s\frac{n!}{(n-s)!}\max_{0\leq j\leq s}\sup|\rho^{(j)}(x)|$$

$$\leq \underbrace{c\,2^{m+1}\max_{0\leq s\leq m}\sup|\rho^{(s)}|}_{c_m}\cdot\frac{n!}{(n-m)!}$$

$$\therefore\ |\langle u_1,\varphi\rangle| = 2^{-n}n! \leq c_m\frac{n!}{(n-m)!}$$

and so $(n-m)! \leq c_m 2^n$.

If n is large this is false, and so u_1 isn't a distribution (for instance we could refer to Stirling's formula $n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$, but or can get it cheaper).

u_2 is a distribution on $\mathbb{R}$ (it's linear and sum is finite on each compact set ...)

u_3 isn't a distribution because it's not linear.

6 (Optional)

(i) Let $\psi \in \mathcal{D}(\mathbb{R})$ be the function from ①. Put $\varphi(x) = c\psi(2x-1)$, $x \in \mathbb{R}$. Then clearly φ is C^∞ and since $\varphi(x) \neq 0$ iff $-1 < 2x-1 < 1$, ie, iff $0 < x < 1$, we have $\operatorname{spt}(\varphi) = [0,1]$. The constant $c > 0$ is chosen so $\int \varphi = 1$. Define

$$g(x) = \int_{-1}^{x} \left(\varphi(-t) - \varphi(t)\right) dt, \quad x \in \mathbb{R}.$$

By FTC, g is C^1 with $g'(x) = \varphi(-x) - \varphi(x)$, hence g is C^∞. Since $\varphi \equiv 0$ on $(-\infty, 0]$ and on $[1, \infty)$ we have $g(x) = 0$ for $x \leq -1$. For $x \geq 1$ we have $g(x) = \int_{-1}^{x} \varphi(-t)\,dt - \int_{\mathbb{R}} \varphi(t)\,dt$ $= \int_{-1}^{0} \varphi(-t)\,dt - 1 = \int_0^1 \varphi - 1 = 0$, hence $\operatorname{spt}(g) \subseteq [-1,1]$. Also $g(0) = \int_{-1}^{0} \left(\varphi(-t) - \varphi(t)\right) dt$ $= \int_{-1}^{0} \varphi(-t)\,dt = \int_0^1 \varphi(t)\,dt = 1$, $g'(0) = \varphi(-0) - \varphi(0)$ $= 0$ and $g^{(k+1)}(x) = (-1)^k \varphi^{(k)}(-x) - \varphi^{(k)}(x)$, so $g^{(k+1)}(0) = \left((-1)^k - 1\right) \varphi^{(k)}(0) = 0$ since $\operatorname{spt}(\varphi) = [0,1]$. (Can also be done using cut-off function from lectures.)

(ii) $g_n(x) = g(\frac{x}{\varepsilon_n}) \frac{a_n x^n}{n!}$, $x \in \mathbb{R}$, is C^∞ by Leibniz and $g_n(x) = 0$ when $|\frac{x}{\varepsilon_n}| \geq 1$, that is, when $|x| \geq \varepsilon_n$. Consequently, $g_n \in \mathcal{D}(\mathbb{R})$ and $\operatorname{spt}(g_n) \subseteq [-\varepsilon_n, \varepsilon_n]$.

$$g_0(0) = g(0) \frac{a_0 0^0}{0!} = a_0, \quad g_n(0) = 0 \text{ for } n \in \mathbb{N}.$$

By Leibniz we get for $k \in \mathbb{N}$:

$$g_n^{(k)}(x) = \sum_{j=0}^{k} \binom{k}{j} g^{(j)}\left(\frac{x}{\varepsilon_n}\right) \varepsilon_n^{-j} \frac{d^{k-j}}{dx^{k-j}}\left(\frac{a_n x^n}{n!}\right),$$

hence $g_n^{(k)}(0) \overset{(i)}{=} \left.\frac{d^k}{dx^k}\right|_{x=0} \left(\frac{a_n x^n}{n!}\right) = a_n f_{k,n}$.

Next, since $\operatorname{spt}(g) \subseteq [-1,1]$ we get for $0 \leq k < n$, $x \in \mathbb{R}$:

$$|g_n^{(k)}(x)| \leq \sum_{j=0}^{k} \binom{k}{j} \left| g^{(j)}\left(\frac{x}{\varepsilon_n}\right) \varepsilon_n^{-j} a_n \frac{x^{n-k+j}}{(n-k+j)!}\right|$$

$$\leq \sum_{j=0}^{k} \binom{k}{j} \sup|g^{(j)}| \varepsilon_n^{-j} \frac{|a_n|}{(n-k+j)!} \varepsilon_n^{n-k+j}$$

$$\leq \max_{0 \leq j < n} \sup|g^{(j)}| \sum_{j=0}^{k} \binom{k}{j} \frac{|a_n|}{(n-k+j)!} \varepsilon_n^{n-k}$$

$$\leq \max_{0 \leq j < n} \sup|g^{(j)}| \sum_{j=0}^{k} \binom{k}{j} \frac{|a_n|}{n!} \varepsilon_n^{n-k}$$

$$= \frac{|a_n|}{n!} 2^k \max_{0 \le j < n} \sup |g^{(j)}| \, \varepsilon_n^{n-k}$$

$$\le \frac{|a_n| 2^n}{n!} \max_{0 \le j < n} \sup |g^{(j)}| \cdot \varepsilon_n \quad \text{provided } \varepsilon_n \le 1.$$

Take $$\varepsilon_n = \frac{2^{-n}}{1 + \frac{|a_n| 2^n}{n!} \max_{0 \le j < n} \sup |g^{(j)}|}$$ to

conclude. □

(iii) With above ε_n, put $f(x) = \sum_{n=0}^{\infty} g_n(x)$, $x \in \mathbb{R}$

Because $|g_n(x)| \le 2^{-n}$ for all x it follows from Weierstrass' M-test that the series is uniformly convergent on $\mathbb{R}$. As each g_n is C^∞, f is in particular continuous.

Assume for some $k \in \mathbb{N}_0$, f is C^k with $$f^{(k)}(x) = \sum_{n=0}^{\infty} g_n^{(k)}(x), \quad x \in \mathbb{R}.$$

Then by (ii) (choice of ε_n): $|g_n^{(k+1)}(x)| \le 2^{-n}$ for $x \in \mathbb{R}$, $n > k+1$, so Weierstrass again gives uniform convergence of $\sum_{n > k+1} g_n^{(k+1)}(x)$ on $\mathbb{R}$, and hence of $\sum_{n=0}^{\infty} g_n^{(k+1)}(x)$ on $\mathbb{R}$.

By a result from prelims f is C^{k+1} with $f^{(k+1)}(x) = \sum_{n=0}^{\infty} g_n^{(k+1)}(x)$ for $x \in \mathbb{R}$.

Thus by induction, f is C^{∞}. Since g_n is supported in $[-\varepsilon_n, \varepsilon_n]$ and $\varepsilon_1 \leq 1$ it follows that f is supported in $[-1,1]$.

Finally, $f^{(n)}(0) = \sum_{k=0}^{\infty} g_k^{(n)}(0) = a_n$ for all $n \in \mathbb{N}_0$.

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7 (optional)

(i) $(h_r * h_s)(x) = \frac{1}{rs} \int_{-\infty}^{\infty} \mathbb{1}_{(0,r)}(x-y)\, \mathbb{1}_{(0,s)}(y)\, dy =$

$\frac{1}{rs} \int_0^s \mathbb{1}_{(0,r)}(x-y)\, dy = \frac{1}{rs} \int_{x-s}^{x} \mathbb{1}_{(0,r)}(t)\, dt =$

$$\frac{1}{rs} \mathcal{L}^1\big((x-s,x) \cap (0,r)\big) = \begin{cases} 0 & x \leq 0 \\ x/rs & 0 < x \leq r \\ 1/s & r < x \leq s \\ \frac{r+s-x}{rs} & s < x \leq r+s \\ 0 & r+s < x \end{cases}$$

so continuous by inspection (in fact, it's Lipschitz continuous with Lipschitz constant $\frac{1}{rs}$)

We also note that $\mathrm{spt}(h_r * h_s) = [0, r+s]$,

and that $0 \leq h_r * h_s \leq \frac{1}{s}$.

(ii) For $x \in \mathbb{R}$, $(h_r * u)(x) = \frac{1}{r}\int_{-\infty}^{\infty} \mathbb{1}_{(0,r)}(x-y)u(y)dy$

$= \frac{1}{r}\int_{x-r}^{x} u(y)\,dy$, and so $h_r * u$ is C^1 by FTC with $(h_r * u)'(x) = \frac{1}{r}(u(x) - u(x-r))$.

But since u is C^k, $h_r * u$ must then be C^{k+1} with $(h_r * u)^{(k+1)}(x) = \frac{1}{r}(u^{(k)}(x) - u^{(k)}(x-r))$.

By inspection we get from $\mathrm{spt}(u) \subseteq [a,b]$ that $\mathrm{spt}(h_r * u) \subseteq [a, b+r]$.

(iii) We have $u_1 = h_{r_0} * h_{r_1}$ is C^0 with $\mathrm{spt}(u_1) \subseteq [0, r_0 + r_1]$ and $0 \leq u_1 \leq \frac{1}{r_0}$, so claim is true for $n=1$. Now assume it's true for $m \leq n \in \mathbb{N}$, any $0 < r_0 \leq r_1 \leq \cdots \leq r_m$

$v_m = \cancel{u_{m+1}}\ h_{r_0} * h_{r_1} * \cdots * h_{r_m}$ is C_c^{m-1} with support in $[0, r_0 + \cdots + r_m]$ and $v_m^{(k)}$ has values in ~~$[0,$~~ $\left[-\frac{2^k}{r_0 \cdots r_k}, \frac{2^k}{r_0 \cdots r_k}\right]$, $0 \leq k < m$

Consider $u_{n+1} = h_{r_0} * \cdots * h_{r_{n+1}} = h_{r_0} * v_n$, where $v_n = h_{r_1} * \cdots * h_{r_{n+1}}$. By induction hypothesis, v_n is C_c^{n-1} with

$\mathrm{spt}(v_n) \subseteq [0, r_1 + \cdots + r_{n+1}]$ and

$$|v_n^{(k)}| \leq \frac{2^k}{r_1 r_2 \cdots r_{k+1}} \quad \text{for } 0 \leq k < n.$$

Now by (ii) we get $u_{n+1} = h_{r_0} * v_n$ is C_c^{k} with $\mathrm{spt}(u_{n+1}) \subseteq [0, r_0 + \cdots + r_{n+1}]$ and

$$u_{n+1}^{(k)}(x) = \frac{v_n^{(k-1)}(x) - v_n^{(k-1)}(x - r_0)}{r_0}.$$

Hence

$$\begin{aligned} |u_{n+1}^{(k)}(x)| &\leq \frac{1}{r_0}\left(|v_n^{(k-1)}(x)| + |v_n^{(k-1)}(x - r_0)|\right) \\ &\leq \frac{1}{r_0}\left(\frac{2^{k-1}}{r_1 \cdots r_k} + \frac{2^{k-1}}{r_1 \cdots r_k}\right) \\ &= \frac{2^k}{r_0 r_1 \cdots r_k} \quad \text{for } k < n+1, \end{aligned}$$

and the claim follows by induction.

(iv) For $m, n \in \mathbb{N}$ and $x \in \mathbb{R}$:

$$|u_{m+n}(x) - u_n(x)| = \left|\int_{-\infty}^{\infty} \big(u_n(x-y) - u_n(x)\big) v(y)\, dy\right|$$

where $v(y) = (h_{r_{n+1}} * \cdots * h_{r_{n+m}})(y)$. Recall that $\int_{-\infty}^{\infty} v = 1$, and $\mathrm{spt}(v) \subseteq [0, r_{n+1} + \cdots + r_{n+m}]$ so using FTC and the bound from (iii)

$$|u_n(x-y)-u_n(x)| = \left| \int_0^1 u_n'(x+ty)\,y\,dt \right|$$

$$\leq \frac{2}{r_0 r_1}|y|,$$

we get $|u_{m+n}(x)-u_n(x)| \leq \int_0^{r_{n+1}+\dots+r_{n+m}} \frac{2}{r_0 r_1}|y|\,v(y)\,dy$

$\leq \frac{2}{r_0 r_1}(r_{n+1}+\dots+r_{n+m})$. It follows that (u_n) is uniformly Cauchy on $\mathbb{R}$, and so $u(x) = \lim_{n\to\infty} u_n(x)$ exists as a continuous function. Since with $v_n = h_{r_1} * \dots * h_{r_n}$ and $u_n = h_{r_0} * v_n$ we have as in (iii)

$$u_n^{(k)}(x) = \frac{v_n^{(k-1)}(x) - v_n^{(k-1)}(x-r_0)}{r_0}, \quad k<n,$$

we can repeat the above argument to see that $(u_n^{(k)})$ is a uniform Cauchy sequence on $\mathbb{R}$, and hence that u is C^k. From (iii) we then deduce that $\mathrm{spt}(u) \subseteq [0,R]$ and

$$|u^{(k)}| \leq \frac{2^k}{r_0 r_1 \cdots r_k}, \quad k \in \mathbb{N}_0. \ \square$$