Distribution Theory and Analysis of PDEs — MT19

Problem Sheet 2

Problem 1. Let $f, g \in \mathrm{C}^1(\mathbb{R})$ and define

$$u(x) = \begin{cases} f(x) & \text{if } x < 0 \\ g(x) & \text{if } x \geq 0. \end{cases}$$

Explain why $u \in \mathscr{D}'(\mathbb{R})$ and calculate the distributional derivative u'. What can you say about the function

$$v(x) = \begin{cases} f(x) & \text{if } x < 0 \\ a & \text{if } x = 0 \\ g(x) & \text{if } x > 0, \end{cases}$$

where $a \in \mathbb{R}$ is a constant that is different from both $f(0)$ and $g(0)$?

Problem 2. (a) Prove that if $f\colon \mathbb{R} \to \mathbb{R}$ is piecewise continuous and $k \in \mathbb{R}$, then the function $u(x,t) = f(x-kt)$, $(x,t) \in \mathbb{R}^2$, is locally integrable on $\mathbb{R}^2$. Conclude that it defines a distribution and show that it satisfies the one-dimensional wave equation:

$$\frac{\partial^2 u}{\partial t^2} = k^2 \frac{\partial^2 u}{\partial x^2}$$

in the sense of distributions on $\mathbb{R}^2$.
(b) Prove that $u(x,y) = \log\bigl(x^2+y^2\bigr)$ is locally integrable on $\mathbb{R}^2$, and that we have

$$\Delta u = 4\pi\delta_0$$

in the sense of distributions on $\mathbb{R}^2$, where δ_0 is the Dirac delta function on $\mathbb{R}^2$ concentrated at the origin.

Problem 3. Let $a > 0$. For each $\varphi \in \mathscr{D}(\mathbb{R})$ we let

$$\langle T_a, \varphi\rangle = \left(\int_{-\infty}^{-a} + \int_a^{\infty}\right) \frac{\varphi(x)}{|x|}\,\mathrm{d}x + \int_{-a}^{a} \frac{\varphi(x)-\varphi(0)}{|x|}\,\mathrm{d}x.$$

Show that T_a hereby is well-defined and that it is a distribution on $\mathbb{R}$.
Now assume that $\varphi \in \mathscr{D}(\mathbb{R})$ satisfies $\varphi(0) = 0$. Show that then

$$\langle T_a, \varphi\rangle = \int_{-\infty}^{\infty} \frac{\varphi(x)}{|x|}\,\mathrm{d}x.$$

What distribution is $T_a - T_b$ for $0 < b < a$?

Problem 4. (a) Let $\alpha \in (-n, \infty)$ and $u_\alpha(x) = |x|^\alpha$ for $x \in \mathbb{R}^n \setminus \{0\}$. Show that u_α is a regular distribution on $\mathbb{R}^n$. (*Hint: Use polar coordinates.*)

(b) For each $r > 0$ we define the r-dilation of a test function $\varphi \in \mathscr{D}(\mathbb{R}^n)$ by the rule

$$(d_r\varphi)(x) = \varphi(rx), \quad x \in \mathbb{R}^n.$$

Extend the r-dilation to distributions $u \in \mathscr{D}'(\mathbb{R}^n)$.
(c) Show that for the distribution u_α defined in (a) we have $d_r u_\alpha = r^\alpha u_\alpha$ for all $r > 0$. We express this by saying that u_α is *homogeneous of degree* α.
(d) Show that the Dirac delta function δ_0 concentrated at the origin $0 \in \mathbb{R}^n$ is homogeneous of degree $-n$.
(e) Let $u \in \mathscr{D}'(\mathbb{R}^n)$ be homogeneous of degree $\beta \in \mathbb{R}$: $d_r u = r^\beta u$ for all $r > 0$. Show that for each $j \in \{1, \dots, n\}$ the distribution $x_j u$ is homogeneous of degree $\beta + 1$ and that the distribution $D_j u$ is homogeneous of degree $\beta - 1$. Finally show that

$$\sum_{j=1}^{n} x_j D_j u = \beta u. \tag{1}$$

This PDE is known as Euler's relation for β-homogeneous distributions.
(f) (Optional) Show that a distribution $u \in \mathscr{D}'(\mathbb{R}^n)$ that satisfies (1) must be homogeneous of degree β.

Problem 5. Show that δ_a, the Dirac delta function concentrated at $a \in \mathbb{R}$, satisfies the equation

$$(x - a)u = 0. \tag{2}$$

Find the general solution $u \in \mathscr{D}'(\mathbb{R})$ to (2). (*Hint: See Corollary 1.10 in the Lecture Notes.*)

Problem 6. (Distribution defined by principal value integral)
Define for each $\varphi \in \mathscr{D}(\mathbb{R})$,

$$\langle \mathrm{pv}\big(\frac{1}{x}\big), \varphi \rangle = \lim_{a \to 0+} \left(\int_{-\infty}^{-a} + \int_{a}^{\infty} \right) \frac{\varphi(x)}{x} \, \mathrm{d}x.$$

(a) Show that hereby $\mathrm{pv}\big(\frac{1}{x}\big) \in \mathscr{D}'(\mathbb{R})$ and that it is homogeneous of order -1 (see Problem 4). Check that

$$\frac{\mathrm{d}}{\mathrm{d}x} \log|x| = \mathrm{pv}\big(\frac{1}{x}\big).$$

(b) Show that $u = \mathrm{pv}\big(\frac{1}{x}\big)$ solves the equation

$$xu = 1 \tag{3}$$

in the sense of $\mathscr{D}'(\mathbb{R})$. What is the general solution $u \in \mathscr{D}'(\mathbb{R})$ to (3)?

Distribution Theory and Analysis of PDEs

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(1) u is piecewise C^1 and so in particular locally integrable on $\mathbb{R}$. We may therefore identify u with the distribution

$$\langle u,\varphi\rangle := \int_{-\infty}^{\infty} u(x)\varphi(x)\,dx\ ,\quad \varphi\in\mathcal{D}(\mathbb{R}).$$

By definition of distributional derivative:

$$\langle u',\varphi\rangle = \langle u,-\varphi'\rangle = -\int_{-\infty}^{\infty} u(x)\varphi'(x)\,dx =$$

$$-\int_{-\infty}^{0} f(x)\varphi'(x)\,dx - \int_{0}^{\infty} g(x)\varphi'(x)\,dx \overset{\text{parts}}{=}$$

$$-\big[f(x)\varphi(x)\big]_{x\to-\infty}^{x=0} + \int_{-\infty}^{0} f'(x)\varphi(x)\,dx - \big[g(x)\varphi(x)\big]_{x=0}^{x\to\infty} + \int_{0}^{\infty} g'(x)\varphi(x)\,dx$$

$$= -f(0)\varphi(0) + g(0)\varphi(0) + \int_{-\infty}^{\infty}\left(f'\mathbb{1}_{(-\infty,0)} + g'\mathbb{1}_{(0,\infty)}\right)\varphi\,dx$$

so $u' = (g(0)-f(0))\delta_0 + f'\mathbb{1}_{(-\infty,0)} + g'\mathbb{1}_{(0,\infty)}$.

Since $u(x) = v(x)$ a.e. it follows that $u=v$ as distributions on $\mathbb{R}$ (or in notation from lectures: $T_u = T_v$) and so $v'=u'$ as distributions on $\mathbb{R}$.

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(2) See Section 2.5 in

[Strichartz: A Guide to Distribution Theory and Fourier Transforms, World Scientific.]

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(3) Well-defined: Let $\varphi \in \mathcal{D}(\mathbb{R})$.

Clearly the first two integrals are well-defined. For the integral over $[-a,a]$ put

$$\Phi(x) = \begin{cases} \dfrac{\varphi(x)-\varphi(0)}{|x|} & , x \neq 0 \\ \varphi'(0) & , x = 0 \end{cases}.$$

Then Φ is piecewise continuous (cont. from the right at 0), hence is (Riemann-) integrable on $[-a,a]$.

Linearity: of T_a now follows from linearity of the integral.

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Fix a compact set $K \subset \mathbb{R}$. Take $\ell > a$ so $K \subset [-\ell, \ell]$. For $\varphi \in \mathcal{D}(K)$:

$$\left| \left(\int_{-\infty}^{-a} + \int_{a}^{\infty} \right) \frac{\varphi(x)}{|x|} dx \right| \leq 2 \int_a^{\ell} \frac{dx}{x} \sup|\varphi|$$

$$= 2 \log \frac{\ell}{a} \sup|\varphi| ,$$

$$\left| \int_{-a}^{a} \frac{\varphi(x) - \varphi(0)}{|x|} dx \right| \overset{\text{FTC}}{=} \left| \int_{-a}^{a} \int_0^1 \varphi'(tx) dt \frac{x}{|x|} dx \right|$$

$$\leq \int_{-a}^{a} \int_0^1 |\varphi'(tx)| dt\, dx$$

$$\leq 2a \sup|\varphi'|$$

$$\therefore \; |\langle T_a, \varphi \rangle| \leq c (\sup|\varphi| + \sup|\varphi'|) ,$$

where $c = 2 \max(a, \log \frac{\ell}{a})$.

Hence $T_a \in \mathcal{D}(\mathbb{R})$ (and is of order at most 1).

If $\varphi \in \mathcal{D}(\mathbb{R})$ with $\varphi(0) = 0$, then $\langle T_a, \varphi \rangle =$

$$\left(\int_{-\infty}^{-a} + \int_a^{\infty} \right) \frac{\varphi(x)}{|x|} dx + \int_{-a}^{a} \frac{\varphi(x) - 0}{|x|} dx = \int_{-\infty}^{\infty} \frac{\varphi(x)}{|x|} dx .$$

If $0 < a < b$, then for $\varphi \in \mathcal{D}(\mathbb{R})$:

$$\langle T_a - T_b, \varphi \rangle = \left[\left(\int_{-\infty}^{-a} + \int_a^{\infty} \right) - \left(\int_{-\infty}^{-b} + \int_b^{\infty} \right) \right] \frac{\varphi(x)}{|x|} dx +$$

$$\left[\int_{-a}^{a} - \int_{-b}^{b}\right] \frac{\varphi(x)-\varphi(0)}{|x|}\,dx =$$

$$\left[\int_{-b}^{-a} + \int_{a}^{b}\right] \frac{\varphi(x)}{|x|}\,dx - \left[\int_{-b}^{-a} + \int_{a}^{b}\right] \frac{\varphi(x)-\varphi(0)}{|x|}\,dx =$$

$$\left[\int_{-b}^{-a} + \int_{a}^{b}\right] \frac{\varphi(0)}{|x|}\,dx = 2\int_{a}^{b} \frac{dx}{x}\,\varphi(0) = 2\log\frac{b}{a}\,\varphi(0)$$

$$\therefore \quad T_a - T_b = 2\log\frac{b}{a}\,\delta_0$$

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If $v=0$, then $u=0$ and any constant c will do. Assume $v\neq 0$. Then we can find $\tilde{\chi}\in\mathcal{D}(\Omega)$ so $\langle v,\tilde{\chi}\rangle\neq 0$. Define $\chi = \dfrac{\tilde{\chi}}{\langle v,\tilde{\chi}\rangle}$. Then $\chi\in\mathcal{D}(\Omega)$ and $\langle v,\chi\rangle = 1$.

Now for $\varphi\in\mathcal{D}(\Omega)$, $\varphi - \langle v,\varphi\rangle\chi\in\mathcal{D}(\Omega)$ and $\langle v,\varphi-\langle v,\varphi\rangle\chi\rangle = 0$, hence also

$0 = \langle u,\varphi-\langle v,\varphi\rangle\chi\rangle = \langle u,\varphi\rangle - \langle v,\varphi\rangle\langle u,\chi\rangle$

and so $u = cv$ for $c = \langle u,\chi\rangle$. $\square$

$$-f(0)\varphi(0) + \int_{-\infty}^{0} f'\varphi\,dx + g(0)\varphi(0) + \int_0^\infty g'\varphi\,dx =$$

$$\langle (g(0)-f(0))\delta_0, \varphi\rangle + \int_{\mathbb{R}} \left(f'\mathbb{1}_{(-\infty,0)} + g'\mathbb{1}_{(0,\infty)}\right)\varphi\,dx$$

so $$u' = (g(0)-f(0))\delta_0 + f'\mathbb{1}_{(-\infty,0)} + g'\mathbb{1}_{(0,\infty)}.$$

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4 (a) $u_\alpha(x) = |x|^\alpha$, $\alpha > -n$. 5/14

u_α is clearly locally integrable in $\mathbb{R}^n\setminus\{0\}$.

Integrating in polar coordinates we get

$$\int_{B_1(0)} |u_\alpha(x)|\,dx = \int_0^1 \int_{|x|=r} |x|^\alpha \, dS_x \, dr \overset{(*)}{=}$$

$$\int_0^1 r^\alpha \cdot \omega_n r^{n-1}\,dr =$$

$$\left[\frac{\omega_n}{n+\alpha} r^{n+\alpha}\right]_{r\to 0+}^{r=1} = \frac{\omega_n}{n+\alpha} < \infty$$

(*) ω_n = surface area of $\partial B_1(0)$ in $\mathbb{R}^n$ = $\dfrac{\mathcal{L}^n(B_1(0))}{n}$

since $\alpha > -n$.

Thus $u_\alpha \in L^1_{loc}(\mathbb{R}^n) \subset \mathcal{D}'(\mathbb{R}^n)$.

(b) For $\varphi,\psi \in \mathcal{D}(\mathbb{R}^n)$ we have

$$\int_{\mathbb{R}^n} d_r\varphi\,\psi\,dx = \int_{\mathbb{R}^n} \varphi(rx)\psi(x)\,dx \overset{\substack{y=rx \\ dy = r^n dx}}{=}$$

$$\int_{\mathbb{R}^n} \varphi(y)\psi\!\left(\tfrac{y}{r}\right)\tfrac{1}{r^n}\,dy = \int_{\mathbb{R}^n} \varphi\, r^{-n} d_{r^{-1}}\psi\,dy$$

an adjoint identity with $r^n d_{\frac{1}{r}} : \mathcal{D}(\mathbb{R}^n) \to \mathcal{D}(\mathbb{R}^n)$ clearly linear and continuous. We may thus define for $u \in \mathcal{D}'(\mathbb{R}^n)$:

$$\langle d_r u, \varphi \rangle \overset{\text{def}}{=} \langle u, \tfrac{1}{r^n} d_{\frac{1}{r}} \varphi \rangle, \quad \varphi \in \mathcal{D}(\mathbb{R}^n).$$

(c) For $\varphi \in \mathcal{D}(\mathbb{R}^n)$: $\langle d_r u_\alpha, \varphi \rangle =$

$$\langle u_\alpha, \tfrac{1}{r^n} d_{\frac{1}{r}} \varphi \rangle = \int_{\mathbb{R}^n} |x|^\alpha \frac{1}{r^n} \varphi\left(\frac{x}{r}\right) dx \underset{dy = \frac{1}{r^n} dx}{\overset{y = \frac{x}{r}}{=}}$$

$$\int_{\mathbb{R}^n} |ry|^\alpha \frac{1}{r^n} \varphi(y) \cdot r^n dy = r^\alpha \int_{\mathbb{R}^n} |y|^\alpha \varphi(y) dy =$$

$r^\alpha \langle u_\alpha, \varphi \rangle$, as required.

(d) $\langle d_r \delta_0, \varphi \rangle = \langle \delta_0, \frac{1}{r^n} d_{\frac{1}{r}} \varphi \rangle = \frac{1}{r^n} \varphi\left(\frac{0}{r}\right) =$

$\frac{1}{r^n} \langle \delta_0, \varphi \rangle$, that is, $d_r \delta_0 = \frac{1}{r^n} \delta_0$ for all $r > 0$.

(e) $\langle d_r(x_j u), \varphi \rangle = \langle x_j u, \frac{1}{r^n} d_{\frac{1}{r}} \varphi \rangle =$

$\langle u, x_j \frac{1}{r^n} d_{\frac{1}{r}} \varphi \rangle$.

Now $x_j \left(\frac{1}{r^n} d_{\frac{1}{r}} \varphi\right)(x) = x_j \frac{1}{r^n} \varphi\left(\frac{x}{r}\right) = \frac{1}{r^{n-1}} (x_j \varphi)\left(\frac{x}{r}\right) =$

$r \frac{1}{r^n} d_{\frac{1}{r}} (x_j \varphi)$ so $\langle u, x_j \frac{1}{r^n} d_{\frac{1}{r}} \varphi \rangle = r \langle u, \frac{1}{r^n} d_{\frac{1}{r}} (x_j \varphi) \rangle$

$= r \langle d_r u, x_j \varphi \rangle = r \cdot r^\beta \langle u, x_j \varphi \rangle = r^{\beta+1} \langle x_j u, \varphi \rangle$

Hence $d_r(x_j u) = r^{\beta+1} u$, as required. $\frac{7}{14}$

$< d_r(D_j u), \varphi > = < D_j u, \frac{1}{r^n} d_{\frac{1}{r}} \varphi > =$

$-< u, D_j \frac{1}{r^n} d_{\frac{1}{r}} \varphi >$. Calculate: $D_j(\frac{1}{r^n} d_{\frac{1}{r}} \varphi)(x) =$

$\frac{1}{r^n} D_j(\varphi(\frac{x}{r})) = \frac{1}{r^{n+1}} (D_j \varphi)(\frac{x}{r}) = \frac{1}{r} \cdot \frac{1}{r^n} d_{\frac{1}{r}} (D_j \varphi)(x)$,

hence $< u, D_j \frac{1}{r^n} d_{\frac{1}{r}} \varphi > = \frac{1}{r} < d_r u, D_j \varphi >$ and

so $< d_r(D_j u), \varphi > = -\frac{1}{r} < d_r u, D_j \varphi > =$

$-\frac{1}{r} r^\beta < u, D_j \varphi > = r^{\beta-1} < D_j u, \varphi >$. Thus

$d_r(D_j u) = r^{\beta-1} D_j u$, as required.

For $\varphi \in \mathcal{D}(\mathbb{R}^n)$ we calculate:

$$< \sum_{j=1}^n x_j D_j u, \varphi > = < u, -\sum_{j=1}^n D_j(x_j \varphi) > =$$

$$< u, -\sum_{j=1}^n (\varphi + x_j D_j \varphi) > = < u, -n\varphi - x \cdot \nabla \varphi >.$$

By above, $\sum_{j=1}^n x_j D_j u$ is homogeneous of degree β, as u is: $d_r u = r^\beta u$ for $r > 0$,

or:

$$< u, r^{-n} d_{r^{-1}} \varphi > = r^\beta < u, \varphi > \quad \text{for } r > 0.$$

Consequently, $\beta < u, \varphi > = \frac{d}{dr}\Big|_{r=1} < u, r^{-n} d_{r^{-1}} \varphi >$

Note

$$\frac{d}{dr}\Big|_{r=1}\left(r^{-n} d_{r^{-1}}\varphi\right)(x) = -n\varphi(x) - \nabla\varphi(x)\cdot x,$$

and for $0<\varepsilon<1$, $0<|t|<\varepsilon$:

$$\Delta_t(x) = \frac{1}{t}\left((1+t)^{-n}\varphi\left(\frac{x}{1+t}\right) - \varphi(x)\right) \xrightarrow[t\to 0]{} -n\varphi(x) - \nabla\varphi(x)\cdot x$$

uniformly in $x\in\mathbb{R}^n$. If we let

$$R = \sup\left\{ \frac{|x|}{1-\varepsilon} : x\in \operatorname{spt}(\varphi)\right\},$$

then $\operatorname{spt}\Delta_t \subset B_R(0)$ for $0<|t|<\varepsilon$, and for $\alpha\in\mathbb{N}_0^n$ we have

$$D^\alpha \Delta_t(x) = \frac{1}{t}\left((1+t)^{-n-|\alpha|}(D^\alpha\varphi)\left(\frac{x}{1+t}\right) - D^\alpha\varphi(x)\right)$$

$$\xrightarrow[t\to 0]{} -(n+|\alpha|)D^\alpha\varphi(x) - \nabla(D^\alpha\varphi)(x)\cdot x$$

uniformly in $x\in\mathbb{R}^n$. Observe that

$$D^\alpha\left(-n\varphi(x) - \nabla\varphi(x)\cdot x\right) = -nD^\alpha\varphi(x) - \sum_{j=1}^n D^\alpha(x_jD_j\varphi(x))$$

and by Leibniz

$$D^\alpha(x_jD_j\varphi(x)) = \sum_{\beta\leq\alpha}\binom{\alpha}{\beta} D^\beta(x_j)\, D^{\alpha-\beta}(D_j\varphi(x)).$$

Since

$$D^\beta(x_j) = \begin{cases} x_j & \text{if } \beta=0 \\ 1 & \text{if } \beta=e_j \\ 0 & \text{else}\end{cases}$$

the above simplifies to

$$x_j D^{\alpha+e_j}\varphi(x) + \binom{\alpha}{e_j} D^{\alpha-e_j}(D_j\varphi(x)) =$$

$$x_j D_j(D^\alpha\varphi)(x) + \alpha_j D^\alpha\varphi(x),$$ and consequently

$$D^\alpha(-n\varphi(x) - \nabla\varphi(x).x) = -nD^\alpha\varphi(x) - \sum_{j=1}^{n}\Big(x_j D_j(D^\alpha\varphi)(x) +$$

$$\alpha_j D^\alpha\varphi(x)\Big) = -(n+|\alpha|)D^\alpha\varphi(x) - x.\nabla(D^\alpha\varphi)(x).$$

We have shown that $\Delta_t(x) \xrightarrow[t\to 0]{} -n\varphi(x) - \nabla\varphi(x).x$ in $\mathcal{D}(\mathbb{R}^n)$, and hence

$$\beta\langle u,\varphi\rangle = \frac{d}{dr}\Big|_{r=1}\langle u, r^{-n}d_{r^{-1}}\varphi\rangle =$$

$$\lim_{t\to 0}\langle u,\Delta_t\rangle = \langle u, \frac{d}{dr}\Big|_{r=1} r^{-n}d_{r^{-1}}\varphi\rangle =$$

$$\langle u, -n\varphi - \nabla\varphi.x\rangle = \langle \sum_{j=1}^{n} x_j \cdot D_j u, \varphi\rangle,$$

as required.

~~(4) See Section 2.5 in Strichartz: A Guide to Distribution Theory and Fourier Transforms, World Scientific.~~

(f) $\boxed{\text{Optional}}$

Assume $u \in \mathcal{D}'(\mathbb{R}^n)$ satisfies the PDE

$x \cdot \nabla u = \beta u.$

To show that $d_r u = r^\beta u$ for $r > 0$ it suffices to show that for each $\varphi \in \mathcal{D}(\mathbb{R}^n)$

$0 < r \longmapsto \langle r^{-\beta} d_r u, \varphi \rangle$ is constant.

Now $\langle r^{-\beta} d_r u, \varphi \rangle = \langle u, \frac{1}{r^{n+\beta}} d_{\frac{1}{r}} \varphi \rangle$

and proceeding as in (e) we see that the function $h(r) = \langle u, \frac{1}{r^{n+\beta}} d_{\frac{1}{r}} \varphi \rangle$, $r > 0$, is differentiable and that

$h'(r) = \langle u, \partial_r \left(\frac{1}{r^{n+\beta}} d_{\frac{1}{r}} \varphi \right) \rangle.$

Now for each $x \in \mathbb{R}^n$, $\partial_r \left(\frac{1}{r^{n+\beta}} \varphi\left(\frac{x}{r}\right) \right) =$

$-(n+\beta) r^{-n-\beta-1} \varphi\left(\frac{x}{r}\right) - r^{-n-\beta-1} \nabla\varphi\left(\frac{x}{r}\right) \cdot \frac{x}{r}$, and so

with $\psi(x) := \varphi\left(\frac{x}{r}\right)$, $\psi \in \mathcal{D}(\mathbb{R}^n)$, $\nabla\psi(x) = \nabla\varphi\left(\frac{x}{r}\right) \frac{1}{r}$

so

$h'(r) = \langle u, -(n+\beta) r^{-n-\beta-1} \psi - r^{-n-\beta-1} \nabla\psi \cdot x \rangle =$

$-(n+\beta) r^{-n-\beta-1} \langle u, \psi \rangle + r^{-n-\beta-1} \langle \operatorname{div}(xu), \psi \rangle =$

$-(n+\beta) r^{-n-\beta-1} \langle u, \psi \rangle + r^{-n-\beta-1} \langle nu + x \cdot \nabla u, \psi \rangle = 0.$

(5) (*) For $\varphi \in \mathcal{D}(\mathbb{R})$:
$\langle x\delta_0, \varphi \rangle = \langle \delta_0, x\varphi \rangle = 0 \cdot \varphi(0) = 0$ so

$x\delta_0 = 0$

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Next, for $\varphi \in \mathcal{D}(\mathbb{R})$ we have by FTC for $x \in \mathbb{R}$ since $\eta(0) = 1$:

$$\varphi(x) - \varphi(0)\eta(x) = \int_0^1 \frac{d}{dt}\left(\varphi(tx) - \varphi(0)\eta(tx)\right) dt$$

$$= \int_0^1 \left(\varphi'(tx) - \varphi(0)\eta'(tx)\right) dt \cdot x.$$

Put $\psi(x) = \int_0^1 \left(\varphi'(tx) - \varphi(0)\eta'(tx)\right) dt$, $x \in \mathbb{R}$. Then ψ is C^∞ and supported in $[-R, R]$, where $R = \sup\{|x| : x \in \operatorname{spt}(\varphi) \cup \operatorname{spt}(\eta)\}$.
The latter follows from $\psi(x) = \frac{\varphi(x) - \varphi(0)\eta(x)}{x} = 0$ for $|x| \geq R$. Thus $\psi \in \mathcal{D}(\mathbb{R})$ and we have $\varphi = \varphi(0)\eta + x\psi$.

Assume $xu = 0$. Then $\langle u, \varphi \rangle = \langle u, \varphi(0)\eta + x\psi \rangle$
$\langle u, \eta \rangle \varphi(0) + \langle xu, \psi \rangle = \langle \langle u, \eta \rangle \delta_0, \varphi \rangle$. Since clearly $c\delta_0$ $(c \in \mathbb{R})$ is a solution we have found GS: $u = c\delta_0$, $c \in \mathbb{R}$.

(*) The above gives the solution to new (5) by translation $0 \to a$: ... substitute x by $x - a$ everywhere above.

~~$m\in\mathbb{N}_0$ we have by Leibniz:~~

$$\cancel{\frac{d^m}{dx^m}(\phi_{x_j}\cdot\rho_{\varepsilon_j}*\varphi) = \sum_{i=0}^{m}\binom{m}{i}\phi_{x_j}^{(i)}\,\rho_{\varepsilon_j}*\varphi^{(m-i)}}$$

~~Since $\phi_{x_j}\equiv 1$ on $\operatorname{spt}(\rho_{\varepsilon_j}*\varphi)$ for large j we have for such j,~~

$$\cancel{\frac{d^m}{dx^m}(\phi_{x_j}\rho_{\varepsilon_j}*\varphi) = \rho_{\varepsilon_j}*\varphi^{(m)} \xrightarrow[j]{} \varphi^{(m)}}$$

~~uniformly. □~~

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⑥ ~~(Optional)~~ 12/14

<u>(a)</u> For $\varphi\in\mathcal{D}(\mathbb{R})$ take $A>0$ so $\operatorname{spt}(\varphi)\subset(-A,A)$ and note that for each $a\in(0,A)$,

$$\left(\int_{-\infty}^{-a}+\int_{a}^{\infty}\right)\frac{\varphi(x)}{x}\,dx = \left(\int_{-A}^{-a}+\int_{a}^{A}\right)\frac{\varphi(x)-\varphi(0)}{x}\,dx$$

Here $\Phi(x) = \begin{cases}\dfrac{\varphi(x)-\varphi(0)}{x}, & 0<|x|\leq A\\ \varphi'(0), & x=0,\end{cases}$.

is continuous, so the improper integral defining $\langle pv(\frac{1}{x}),\varphi\rangle$ is well-defined. It's then clear that $pv(\frac{1}{x}):\mathcal{D}(\mathbb{R})\to\mathcal{D}(\mathbb{R})$ is linear, and since for $\varphi\in\mathcal{D}(-A,A)$ we have that

$$|\langle pv(\tfrac{1}{x}),\varphi\rangle| \leq \max|\varphi'|$$

it follows that $\mathrm{pv}(\frac{1}{x}) \in \mathcal{D}'(\mathbb{R})$ (of order at most 1). For $r>0$ we have

$$< d_r\, \mathrm{pv}(\tfrac{1}{x}), \varphi > = < \mathrm{pv}(\tfrac{1}{x}), \tfrac{1}{r} d_{\frac{1}{r}} \varphi > =$$

$$\lim_{a \to 0+} \left(\int_{-\infty}^{-a} + \int_{a}^{\infty} \right) \frac{\frac{1}{r}\varphi(\frac{x}{r})}{x}\, dx \overset{\substack{y=\frac{x}{r} \\ dy = \frac{dx}{r}}}{=}$$

$$\lim_{a \to 0+} \left(\int_{-\infty}^{-\frac{a}{r}} + \int_{\frac{a}{r}}^{\infty} \right) \frac{\varphi(y)}{ry}\, dy = r^{-1} < \mathrm{pv}(\tfrac{1}{x}), \varphi >$$

showing that $d_r\, \mathrm{pv}(\frac{1}{x}) = r^{-1}\, \mathrm{pv}(\frac{1}{x})$.

For $\varphi \in \mathcal{D}(\mathbb{R})$: $< \frac{d}{dx} \log|x|, \varphi > = \langle \log|x|, -\varphi' \rangle =$

$$-\int_{-\infty}^{\infty} \log|x|\, \varphi'(x)\, dx = -\lim_{\varepsilon \to 0+} \left(\int_{-\frac{1}{\varepsilon}}^{-\varepsilon} + \int_{\varepsilon}^{\frac{1}{\varepsilon}} \right) \log|x|\, \varphi'(x)\, dx$$

$$\overset{\text{parts}}{=} -\lim_{\varepsilon \to 0+} \Big\{ \Big[\log|x| (\varphi(x) - \varphi(0)) \Big]_{x=-\frac{1}{\varepsilon}}^{x=-\varepsilon} - \int_{-\frac{1}{\varepsilon}}^{-\varepsilon} \frac{\varphi(x)-\varphi(0)}{x}\, dx$$

$$+ \Big[\log|x| (\varphi(x) - \varphi(0)) \Big]_{x=\varepsilon}^{x=\frac{1}{\varepsilon}} - \int_{\varepsilon}^{\frac{1}{\varepsilon}} \frac{\varphi(x)-\varphi(x)}{x}\, dx \Big\} =$$

$$\lim_{\varepsilon \to 0+} \left(\int_{-\frac{1}{\varepsilon}}^{-\varepsilon} + \int_{\varepsilon}^{\frac{1}{\varepsilon}} \right) \frac{\varphi(x)-\varphi(0)}{x}\, dx =$$

$$\lim_{\varepsilon \to 0+} \left(\int_{-\frac{1}{\varepsilon}}^{-\varepsilon} + \int_{\varepsilon}^{\frac{1}{\varepsilon}} \right) \frac{\varphi(x)}{x}\, dx = < \mathrm{pv}(\tfrac{1}{x}), \varphi >. \quad \square$$

<u>Note</u> $pr(\frac{1}{x})$ has order 1 : Assume not, then for some constant c we have

$$|\langle pr(\tfrac{1}{x}), \varphi\rangle| \leq c \max|\varphi| \quad \text{for } \varphi \in \mathcal{D}[-1,1].$$

Take $\varphi = \rho_\varepsilon * \mathbb{1}_{[\varepsilon, \frac{1}{2}]}$ for $0<\varepsilon<\frac{1}{2}$. Then

$$c = c\max|\varphi| \geq |\langle pr(\tfrac{1}{x}), \varphi\rangle| = \lim_{a\to 0+} \int_a^\infty \frac{\varphi(x)}{x}\,dx$$

$$= \int_0^\infty \frac{\varphi(x)}{x}\,dx > \int_{2\varepsilon}^{\frac{1}{2}-\varepsilon} \frac{dx}{x} = \log\frac{\frac{1}{2}-\varepsilon}{2\varepsilon} \ \lightning$$

<u>(b)</u> For $\varphi \in \mathcal{D}(\mathbb{R})$ we calculate:

$$\langle x\, pr(\tfrac{1}{x}), \varphi\rangle = \langle pr(\tfrac{1}{x}), x\varphi\rangle =$$

$$\lim_{a\to 0+} \left(\int_{-\infty}^{-a} + \int_a^\infty\right) \frac{x\varphi(x)}{x}\,dx = \int_{-\infty}^{\infty} \varphi(x)\,dx,$$

hence $x\, pr(\frac{1}{x}) = 1$.

Since the equation (2) is linear we get GS from ⑤: $u = pr(\frac{1}{x}) + c\delta_0$, $c \in \mathbb{R}$.