Problem Sheet 3

Problem 1. Find the general solutions to the ODEs

(i)

$$y'' + 2y' + y = 1$$

(ii)

$$y'' + 2y' + y = H$$

(iii)

$$y'' + 2y' + y = \delta_0$$

in $\mathscr{D}'(\mathbb{R})$, where H is Heaviside's function and δ_0 is Dirac's delta-function at 0. What are the classical solutions to (i) and (ii)?

Problem 2. The principal logarithm is defined on the cut plane $\mathbb{C} \setminus (-\infty, 0]$ as

$$\mathrm{Log} z := \log|z| + \mathrm{i}\mathrm{Arg}(z), \quad \mathrm{Arg}(z) \in (-\pi, \pi).$$

Define $\mathrm{Log}(x + \mathrm{i}0)$ and $\mathrm{Log}(x - \mathrm{i}0)$ for each $\varphi \in \mathscr{D}(\mathbb{R})$ by the rules

$$\langle \mathrm{Log}(x \pm \mathrm{i}0), \varphi \rangle := \lim_{\varepsilon \searrow 0} \int_{-\infty}^{\infty} \mathrm{Log}(x \pm \mathrm{i}\varepsilon)\varphi(x)\, \mathrm{d}x.$$

(a) Show that $\mathrm{Log}(x \pm \mathrm{i}0)$ hereby are distributions on $\mathbb{R}$.

Now let $k \in \mathbb{N}$ and define the distributions $(x + \mathrm{i}0)^{-k}$ and $(x - \mathrm{i}0)^{-k}$ as

$$(x \pm \mathrm{i}0)^{-k} := \frac{(-1)^{k-1}}{(k-1)!} \frac{\mathrm{d}^k}{\mathrm{d}x^k} \mathrm{Log}(x \pm \mathrm{i}0) \quad \text{in } \mathscr{D}'(\mathbb{R}).$$

(b) Show that for each $\varphi \in \mathscr{D}(\mathbb{R})$ with $\varphi^{(j)}(0) = 0$ for $j \in \{0, \dots, k\}$ we have

$$\left\langle (x \pm \mathrm{i}0)^{-k}, \varphi \right\rangle = \int_{-\infty}^{\infty} \frac{\varphi(x)}{x^k}\, \mathrm{d}x.$$

(c) Prove that $\mathrm{Log}(x + \mathrm{i}0) - \mathrm{Log}(x - \mathrm{i}0) = 2\pi\mathrm{i}\tilde{H}$, where H is the Heaviside function. Deduce the *Plemelj-Sokhotsky jump relations*:

$$(x + \mathrm{i}0)^{-k} - (x - \mathrm{i}0)^{-k} = 2\pi\mathrm{i}\frac{(-1)^k}{(k-1)!}\delta_0^{(k-1)},$$

where δ_0 is Dirac's delta-function on $\mathbb{R}$ concentrated at 0.

(d) Show that

$$x(x \pm \mathrm{i}0)^{-1} = 1 \quad \text{in } \mathscr{D}'(\mathbb{R}).$$

Deduce that

$$(x+\mathrm{i}0)^{-1}(x\delta_0) = 0 \neq \delta_0 = \Big((x+\mathrm{i}0)^{-1}x\Big)\delta_0.$$

Next, show, for instance by using the differential operator $x\frac{\mathrm{d}}{\mathrm{d}x}$ on the case $k=1$ iteratively, that

$$x^k(x \pm \mathrm{i}0)^{-k} = 1 \quad \text{in } \mathscr{D}'(\mathbb{R})$$

holds for each $k \in \mathbb{N}$.

Problem 3. Let $g \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R})$ and assume that g is T periodic for some $T > 0$: $g(x+T) = g(x)$ holds for almost all $x \in \mathbb{R}$. Define for each $j \in \mathbb{N}$ the function

$$g_j(x) = g(jx), \quad x \in (0,1).$$

Prove that

$$g_j \to \frac{1}{T}\int_0^T g \,\mathrm{d}x \quad \text{in} \quad \mathscr{D}'(0,1) \quad \text{as} \quad j \to \infty.$$

Problem 4. Let $\theta \in \mathscr{D}(\mathbb{R})$.
(i) Explain how the convolution $\theta * u$ is defined for a general distribution $u \in \mathscr{D}'(\mathbb{R})$.
(ii) Prove that $\theta * u \in \mathrm{C}^\infty(\mathbb{R})$ when $u \in \mathscr{D}'(\mathbb{R})$.
(iii) Let $\left(\rho_\varepsilon\right)_{\varepsilon>0}$ be the standard mollifier on $\mathbb{R}$. Show that for a general distribution $u \in \mathscr{D}'(\mathbb{R})$ we have that

$$\rho_\varepsilon * u \to u \text{ in } \mathscr{D}'(\mathbb{R}) \text{ as } \varepsilon \searrow 0.$$

(iv) Show that for each $u \in \mathscr{D}'(\mathbb{R})$ we can find a sequence (u_j) in $\mathscr{D}(\mathbb{R})$ such that

$$u_j \to u \text{ in } \mathscr{D}'(\mathbb{R}) \text{ as } j \to \infty.$$

Problem 5. Let

$$p(\partial) = \sum_{|\alpha|\le k} c_\alpha \partial^\alpha \quad (k \in \mathbb{N} \text{ and } c_\alpha \in \mathbb{C})$$

be a partial differential operator on $\mathbb{R}^n$ in the usual multi-index notation. For an open subset Ω of $\mathbb{R}^n$ and $u \in \mathscr{D}'(\Omega)$ show that the supports always obey the rule:

$$\operatorname{supp}\big(p(\partial)u\big) \subseteq \operatorname{supp}(u).$$

Give an example of a distribution $v \in \mathscr{D}'(\mathbb{R})$ such that the distributional derivative $v' \neq 0$ has compact support, but v itself hasn't.

Next, show that also the singular supports satisfy the rule

$$\operatorname{sing.supp}\big(p(\partial)u\big) \subseteq \operatorname{sing.supp}(u)$$

and give an example of a distribution $u \in \mathscr{D}'(\mathbb{R}^2)$ and a partial differential operator $p(\partial)$ so

$$\text{sing.supp}(u) = \mathbb{R}^2 \text{ and } \text{sing.supp}\big(p(\partial)u\big) = \emptyset.$$

Problem 6. (Optional)
Let $F\colon \mathbb{C} \to \mathbb{C}$ be an entire function that is not identically zero. Explain why the formula $f = \log|F|$ defines a distribution on $\mathbb{C}$.
Prove that its distributional Laplacian equals

$$\Delta f = \sum_{j \in J} 2\pi m_j \delta_{z_j}$$

where $\{z_j : j \in J\}$ are the distinct zeros for F and $\{m_j : j \in J\}$ their multiplicities.
[*Hint: Use the Cauchy-Riemann operators to calculate the Laplacian.*]

(1) (i) $y'' + 2y' + y = 1$ in $\mathcal{D}'(\mathbb{R})$

The characteristic equation has double root -1 so we factorize

$$\frac{d^2}{dx^2} + 2\frac{d}{dx} + I = \left(\frac{d}{dx} + I\right)^2 .$$

If $y \in \mathcal{D}'(\mathbb{R})$ satisfies $y'' + 2y' + y = 0$, then if $z = y' + y$ we have $z' + z = 0$. Now $e^x \in C^\infty(\mathbb{R})$ is an integrating factor, so by Leibniz $0 = e^x(z' + z) = (e^x z)'$.

From the constancy theorem $e^x z = c_1$, $c_1 \in \mathbb{C}$, that is, $z = c_1 e^{-x}$. We then have $y' + y = c_1 e^{-x}$, or: $c_1 = e^x(y' + y) = (e^x y)'$, that is, $0 = (e^x y - c_1 x)'$. By the constancy theorem, $e^x y - c_1 x = c_2$, $c_2 \in \mathbb{C}$, so $y = (c_1 x + c_2)e^{-x}$. We check that functions of this form also satisfies the homogeneous ODE $y'' + 2y' + y = 0$, so that this must be its GS. Clearly $y = 1$ is a PI so GS to (i) in $\mathcal{D}'(\mathbb{R})$ is $1 + (c_1 x + c_2)e^{-x}$, $c_1, c_2 \in \mathbb{C}$.

This is also the GS in the classical sense.

(ii) $y'' + 2y' + y = H$ in $\mathcal{D}'(\mathbb{R})$

We only need to find a PI in $\mathcal{D}'(\mathbb{R})$.
From (i) we consider

$$y = \begin{cases} 0 & , x<0 \\ (Ax+B)e^{-x}+1, & x>0 \end{cases} = \left((Ax+B)e^{-x}+1\right)H$$

We check that this function is differen= tiable when $A=B=-1$:

$y = (1-(x+1)e^{-x})H$. By Leibniz
$y' = xe^{-x}H$ and $y'' = (1-x)e^{-x}H$
and we check $y'' + 2y' + y = H$ in $\mathcal{D}'(\mathbb{R})$.
Thus GS in $\mathcal{D}'(\mathbb{R})$ is: $(c_1x+c_2)e^{-x} + (1-(x+1)e^{-x})H$,
where $c_1, c_2 \in \mathbb{C}$.

GS in classical sense is $\emptyset$ (because classical derivatives have the intermediate value property and H doesn't — or check that none of the above solutions are twice diff.)

(iii) $y'' + 2y' + y = \delta_0$ in $\mathcal{D}'(\mathbb{R})$.
We only need a PI: Diff. (ii) we get (iii) so our PI is $xe^{-x}H$.

and GS in $\mathcal{D}'(\mathbb{R})$ is therefore

$$y = xe^{-x}H(x) + (c_1 x + c_2)e^{-x}, \quad c_1, c_2 \in \mathbb{C}.$$

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(2)

<u>(a)</u> $\mathrm{Log}(x \pm i0)$ are regular distributions on $\mathbb{R}$ since we have for $\varphi \in \mathcal{D}(\mathbb{R})$:

$$\langle \mathrm{Log}(x \pm i0), \varphi \rangle = \int_{-\infty}^{\infty} (\log|x| \pm i\pi H(-x))\varphi(x)\,dx,$$

where $\log|\cdot| \pm i\pi \tilde{H} \in L^1_{loc}(\mathbb{R})$.

<u>(b)</u> Let $\varphi \in \mathcal{D}(\mathbb{R})$ with $\varphi^{(j)}(0) = 0$ for $j = 0, \dots, k$.

Then $\langle (x \pm i0)^{-k}, \varphi \rangle = \frac{(-1)^{k-1}}{(k-1)!} \langle \mathrm{Log}(x \pm i0), (-1)^k \varphi^{(k)} \rangle$

$$= -\frac{1}{(k-1)!} \int_{-\infty}^{\infty} \mathrm{Log}(x \pm i0)\varphi^{(k)}(x)\,dx =$$

$$-\frac{1}{(k-1)!}\left(\left(\int_{-\infty}^{0} + \int_{0}^{\infty}\right) \log|x|\,\varphi^{(k)}(x)\,dx \pm i\pi \int_{-\infty}^{0} \varphi^{(k)}(x)\,dx \right)$$

FTC + parts

$$= -\frac{1}{(k-1)!}\left(\left[\log|x|\,\varphi^{(k-1)}(x)\right]_{x \to -\infty}^{x \nearrow 0} + \left[\log|x|\,\varphi^{(k-1)}(x)\right]_{x \searrow 0}^{x \to \infty} \right.$$

$$\left. - \left(\int_{-\infty}^{0} + \int_{0}^{\infty}\right) \frac{1}{x} \varphi^{(k-1)}(x)\,dx \pm i\pi \left[\varphi^{(k-1)}(x)\right]_{x \to -\infty}^{x \nearrow 0} \right)$$

$$= \frac{1}{(k-1)!}\left(\int_{-\infty}^{0} + \int_{0}^{\infty}\right) \frac{1}{x} \varphi^{(k-1)}(x)\,dx \underset{\text{parts } k-1 \text{ times}}{=}$$

$$\int_{-\infty}^{\infty} \frac{\varphi(x)}{x^k}\,dx \quad , \text{ as required.}$$

(c) From the formula in (a) we get

$\mathrm{Log}(x+i0) - \mathrm{Log}(x-i0) = 2\pi i H(-x)$; hence

$\mathrm{Log}(x+i0) - \mathrm{Log}(x-i0) = 2\pi i \tilde{H}$.

Differentiating k times in the sense of distributions and multiplying by $\frac{(-1)^{k-1}}{(k-1)!}$ we get: $(x+i0)^{-k} - (x-i0)^{-k} =$

$$\frac{(-1)^{k-1}}{(k-1)!} \frac{d^k}{dx^k}\Big(\mathrm{Log}(x+i0) - \mathrm{Log}(x-i0)\Big) = 2\pi i \frac{(-1)^{k-1}}{(k-1)!}\frac{d^k}{dx^k}\tilde{H}$$

$$= 2\pi i \frac{(-1)^{k-1}}{(k-1)!} \frac{d^{k-1}}{dx^{k-1}}(-\delta_0) = 2\pi i \frac{(-1)^k}{(k-1)!}\delta_0^{(k-1)} ,$$

as required.

(d) For $\varphi \in \mathcal{D}(\mathbb{R})$ we have

$$\langle x(x\pm i0)^{-1}, \varphi\rangle = \langle \frac{d}{dx}\mathrm{Log}(x\pm i0), x\varphi\rangle$$

$$= -\int_{-\infty}^{\infty} \log|x|\,(x\varphi)'\,dx \mp i\pi \int_{-\infty}^{0} (x\varphi)'\,dx$$

$$\overset{FTC}{=} -\Big(\int_{-\infty}^{0} + \int_{0}^{\infty}\Big) \log|x|\,(x\varphi)'\,dx \overset{\text{parts}}{=} \int_{-\infty}^{\infty} \frac{1}{x}x\varphi\,dx$$

$$= \int_{-\infty}^{\infty} \varphi\,dx$$

so we have $x(x \pm i0)^{-1} = 1$ in $\mathcal{D}'(\mathbb{R})$.

Deduction is clear (and so one must be careful with the associative rule).

From $x(x \pm i0)^{-1} = 1$ in $\mathcal{D}'(\mathbb{R})$ we get by applying $x\frac{d}{dx}$:

$$0 = x\frac{d}{dx}\left(x(x \pm i0)^{-1}\right) \overset{\text{Leibniz}}{=}$$

$$x\left((x \pm i0)^{-1} + x\left(-(x \pm i0)^{-2}\right)\right) =$$

$$x(x \pm i0)^{-1} - x^2(x \pm i0)^{-2} = 1 - x^2(x \pm i0)^{-2},$$

hence $x^2(x \pm i0)^{-2} = 1$ in $\mathcal{D}'(\mathbb{R})$.

Now suppose that $x^k(x \pm i0)^{-k} = 1$ in $\mathcal{D}'(\mathbb{R})$ for some $k \in \mathbb{N}$. Then we find

$$0 = x\frac{d}{dx}\left(x^k(x \pm i0)^{-k}\right) \overset{\text{Leibniz}}{=}$$

$$x\left(kx^{k-1}(x \pm i0)^{-k} + x^k\left(-k(x \pm i0)^{-k-1}\right)\right) =$$

$$kx^k(x \pm i0)^{-k} - kx^{k+1}(x \pm i0)^{-k-1} =$$

$k - kx^{k+1}(x \pm i0)^{-k-1}$ and induction concludes the proof.

(3) Put $\langle g \rangle = \frac{1}{T}\int_0^T g(t)\,dt$ and $G_j(x) = \frac{1}{j}\int_0^{jx} (g(t) - \langle g \rangle)\,dt$.

Then G_j is continuous so in particular a regular distribution. From the lecture notes we get $G_j' = g_j - \langle g \rangle$ in $\mathcal{D}'(0,1)$.

Or do the calculation with Fubini and FTC again: for $\varphi \in \mathcal{D}(0,1)$

$$\langle G_j', \varphi \rangle = -\langle G_j, \varphi' \rangle = -\frac{1}{j}\int_0^1 \int_0^{jx} (g(t) - \langle g \rangle)\,dt\,\varphi'(x)\,dx$$

$$\overset{\text{Fubini}}{=} -\frac{1}{j}\int_0^\infty \int_0^1 \mathbb{1}_{(0,jx)}(t)\,(g(t) - \langle g \rangle)\,\varphi'(x)\,dx\,dt$$

$$= -\frac{1}{j}\int_0^j \int_{\frac{t}{j}}^1 \varphi'(x)\,dx\,(g(t) - \langle g \rangle)\,dt$$

$$\overset{\text{FTC}}{=} -\frac{1}{j}\int_0^j \left(\varphi(1) - \varphi\left(\tfrac{t}{j}\right)\right)(g(t) - \langle g \rangle)\,dt$$

$$= \frac{1}{j}\int_0^j (g(t) - \langle g \rangle)\,\varphi\left(\tfrac{t}{j}\right)dt.$$

$$\overset{s=\frac{t}{j}}{=} \int_0^1 (g(js) - \langle g \rangle)\,\varphi(s)\,ds;$$

Now $\int_0^1 |G_j(x)|\,dx = \frac{1}{j}\int_0^1 \left| \int_0^{jx} (g(t) - \langle g \rangle)\,dt \right| dx$

$$= \frac{1}{j}\sum_{k=0}^{\lceil \frac{j}{T} \rceil - 2} \int_{k\frac{T}{j}}^{(k+1)\frac{T}{j}} \left| \int_0^{jx} (g(t) - \langle g \rangle)\,dt \right| dx + \frac{1}{j}\int_{(\lceil \frac{j}{T} \rceil - 1)\frac{T}{j}}^1 | \text{—}''\text{—} |\,dx$$

$$\overset{g\ T\text{-periodic}}{=} \frac{1}{j} \sum_{k=0}^{\lceil \frac{j}{T} \rceil - 2} \int_{k\frac{T}{j}}^{(k+1)\frac{T}{j}} \left| \int_{kT}^{jx} (g(t) - \langle g \rangle)\, dt \right| dx$$

$$+ \frac{1}{j} \int_{(\lceil \frac{j}{T} \rceil - 1)\frac{T}{j}}^{1} \left| \int_{(\lceil \frac{j}{T} \rceil - 1)T}^{jx} (g(t) - \langle g \rangle)\, dt \right| dx$$

$$\leq \frac{1}{j} \sum_{k=0}^{\lceil \frac{j}{T} \rceil - 2} \int_{k\frac{T}{j}}^{(k+1)\frac{T}{j}} \int_{kT}^{(k+1)T} |g - \langle g \rangle|\, dt\, dx$$

$$+ \frac{1}{j} \int_{(\lceil \frac{j}{T} \rceil - 1)\frac{T}{j}}^{1} \int_{(\lceil \frac{j}{T} \rceil - 1)T}^{j} |g - \langle g \rangle|\, dt\, dx$$

$$\overset{g\ T\text{-periodic}}{=} \frac{1}{j} \sum_{k=0}^{\lceil \frac{j}{T} \rceil - 2} \frac{T}{j} \int_0^T |g - \langle g \rangle|\, dt$$

$$+ \frac{1}{j} \left(1 - \left(\lceil \tfrac{j}{T} \rceil - 1 \right) \frac{T}{j} \right) \int_0^T |g - \langle g \rangle|\, dt$$

$$= \left(\frac{T}{j^2} \left(\lceil \tfrac{j}{T} \rceil - 1 \right) + \frac{1}{j} \left(1 - \lceil \tfrac{j}{T} \rceil \frac{T}{j} + \frac{T}{j} \right) \right) \int_0^T |g - \langle g \rangle|\, dt$$

$$\leq \left(\frac{1}{j} + \frac{T}{j^2} \right) \int_0^T |g - \langle g \rangle|\, dt \longrightarrow 0 \text{ as } j \to \infty.$$

In particular, $G_j \to 0$ in $\mathcal{D}'(0,1)$ as $j \to \infty$ and by continuity of differentiation in $\mathcal{D}'(0,1)$ also $g_j - \langle g \rangle = G_j' \to 0$ in $\mathcal{D}'(0,1)$ as $j \to \infty$.

(4) Let $\theta \in \mathcal{D}(\mathbb{R})$.

(i) For $\varphi, \psi \in \mathcal{D}(\mathbb{R})$ a calculation gives:

$$\int_{\mathbb{R}} (\theta * \psi)(x)\varphi(x)\,dx = \int_{\mathbb{R}} \psi(x)(\tilde{\theta} * \varphi)(x)\,dx$$

and that the map $\mathcal{D}(\mathbb{R}) \ni \varphi \mapsto \tilde{\theta} * \varphi \in \mathcal{D}(\mathbb{R})$ is linear and $\mathcal{D}(\mathbb{R})$-continuous. We may therefore define $\theta * u$ for $u \in \mathcal{D}'(\mathbb{R})$ by the rule

$$\langle \theta * u, \varphi \rangle := \langle u, \tilde{\theta} * \varphi \rangle, \ \varphi \in \mathcal{D}(\mathbb{R}).$$

We have $\theta * u \in \mathcal{D}'(\mathbb{R})$ well-defined by the adjoint identity scheme. Also note that

$$\mathcal{D}'(\mathbb{R}) \ni u \longmapsto \theta * u \in \mathcal{D}'(\mathbb{R})$$

is linear and $\mathcal{D}'(\mathbb{R})$ - continuous.

(ii) For each $\varphi \in \mathcal{D}(\mathbb{R})$ and $x \in \mathbb{R}$:

$$(\tilde{\theta} * \varphi)(x) = \int_{\mathbb{R}} \theta(y-x)\varphi(y)\,dy.$$

For each fixed $x \in \mathbb{R}$, $y \mapsto \theta(y-x)\varphi(y)$ is uniformly continuous and compactly supported so the integral defining $(\tilde{\theta} * \varphi)(x)$ can be obtained as a limit of Riemann sums. We calculate using dyadic grids: for $k \in \mathbb{N}$,

$$\mathbb{R} = \bigcup_{c \in 2^{-k}\mathbb{Z}} (c, c+2^{-k}] = \bigcup_{j \in \mathbb{Z}} (j2^{-k}, (j+1)2^{-k}]$$

whereby

$$\int_{\mathbb{R}} \theta(y-x)\varphi(y)\,dy = \sum_{j\in\mathbb{Z}} \int_{j2^{-k}}^{(j+1)2^{-k}} \theta(y-x)\varphi(y)\,dy.$$

Put $R_k(x) := \sum_{j\in\mathbb{Z}} 2^{-k}\theta(j2^{-k}-x)\varphi(j2^{-k})$, $x\in\mathbb{R}$.

This is a Riemann sum for the integral defining $(\tilde{\theta}*\varphi)(x)$ and because φ has a compact support the sum is finite. Take $a>0$ so large that $\mathrm{supp}(\theta), \mathrm{supp}(\varphi) \subset [-a,a]$. It follows that $R_k \in C^\infty(\mathbb{R})$ and $\mathrm{supp}(R_k) \subset [-2a,2a]$, so $R_k \in \mathcal{D}(\mathbb{R})$. For each $m\in\mathbb{N}_0$,

$$R_k^{(m)}(x) = \sum_{j\in\mathbb{Z}} (-1)^m 2^{-k}\theta^{(m)}(j2^{-k}-x)\varphi(j2^{-k})$$

and since $y \mapsto (-1)^m\theta^{(m)}(y-x)\varphi(y)$ is uniformly continuous and compactly supported,

$$R_k^{(m)}(x) \longrightarrow (-1)^m \int_{\mathbb{R}} \theta^{(m)}(y-x)\varphi(y)\,dy = (\tilde{\theta}*\varphi)^{(m)}(x)$$

uniformly in $x\in\mathbb{R}$ as $k\to\infty$. Consequently $R_k \longrightarrow \tilde{\theta}*\varphi$ in $\mathcal{D}(\mathbb{R})$ as $k\to\infty$, and so using that u is $\mathcal{D}(\mathbb{R})$-continuous and linear:

$$\langle u, \tilde{\theta}*\varphi\rangle = \lim_k \langle u, R_k\rangle = \lim_k \sum_{j\in\mathbb{Z}} 2^{-k}\langle u, \theta(j2^{-k}-\cdot)\rangle \times \varphi(j2^{-k})$$

Now $y \mapsto \theta(y-x)$ (that is, $\tau_{-x}\tilde{\theta}$) belongs to $\mathcal{D}(\mathbb{R})$ so

$$h(x) := \langle u, \tau_{-x}\tilde{\theta} \rangle, \quad x \in \mathbb{R},$$

is well-defined. Since $x \mapsto \tau_{-x}\tilde{\theta}$ is continuous in the sense $\tau_{-x}\tilde{\theta} \to \tau_{-x_0}\tilde{\theta}$ in $\mathcal{D}(\mathbb{R})$ as $x \to x_0$ (simply note:

$$\mathrm{supp}(\tau_{-x}\tilde{\theta}) = x - \mathrm{supp}(\theta) \subseteq [x-a, x+a]$$
$$\subset [x_0-1-a, x_0+1+a]$$

provided $|x-x_0|<1$ and for $m \in \mathbb{N}_0$,

$$\frac{d^m}{dx^m} \tau_{-x}\tilde{\theta}(y) = (-1)^m \theta^{(m)}(y-x) \to (-1)^m \theta^{(m)}(y-x_0)$$

uniformly in $y \in \mathbb{R}$ as $x \to x_0$) we get by the $\mathcal{D}(\mathbb{R})$-continuity of u that h is continuous. But then $x \mapsto \langle u, \theta(x-\cdot)\rangle \varphi(x)$ is uniformly continuous and compactly supported so

$$\lim_k \sum_{j \in \mathbb{Z}} 2^{-k} \langle u, \theta(j2^{-k} - \cdot)\rangle \varphi(j2^{-k}) = \int_{\mathbb{R}} \langle u, \theta(x-\cdot)\rangle \varphi(x)\,dx$$

Thus $\theta * u \in C(\mathbb{R})$ and $(\theta * u)(x) = \langle u, \theta(x-\cdot)\rangle$, $x \in \mathbb{R}$. Next we check that $\theta * u$ is differentiable with

$(\theta * u)' = \theta' * u$: this follows because the difference quotients

$$\frac{\Delta_h \tilde{\theta}}{h}(\cdot) = \frac{\tilde{\theta}(\cdot + h) - \check{\theta}(\cdot)}{h} \xrightarrow[h \to 0]{} \tilde{\theta}' \text{ in } \mathcal{D}(\mathbb{R})$$

(see Example 2.17 in LN). Since we may apply this argument to $\theta^{(m)}$ instead of θ for each $m \in \mathbb{N}$ it follows by induction that $\theta * u \in C^m(\mathbb{R})$ with $(\theta * u)^{(m)} = \theta^{(m)} * u$ for each $m \in \mathbb{N}$.

(Likewise, since from Example 4.8 in LN we have $\frac{\Delta_h u}{h} \xrightarrow[h \to 0]{} u'$ in $\mathcal{D}'(\mathbb{R})$ one gets by induction that also $(\theta * u)^{(m)} = \theta * u^{(m)}$.)

<u>(iii)</u> Fix $\varphi \in \mathcal{D}(\mathbb{R})$ and consider

$$\langle \rho_\varepsilon * u, \varphi \rangle = \langle u, \tilde{\rho}_\varepsilon * \varphi \rangle = \langle u, \rho_\varepsilon * \varphi \rangle.$$

Since φ in particular is uniformly continuous we have

$$\rho_\varepsilon * \varphi \xrightarrow[\varepsilon \to 0]{} \varphi \quad \text{uniformly on } \mathbb{R}.$$

If $\operatorname{supp}(\varphi) \subset [-a,a]$, then for $0<\varepsilon<1$,
$\operatorname{supp}(\rho_\varepsilon * \varphi) \subset [-a-\varepsilon, a+\varepsilon] \subset [-a-1, a+1]$.
For each $m \in \mathbb{N}$, $\varphi^{(m)}$ is uniformly continuous, so $\rho_\varepsilon * \varphi^{(m)} \xrightarrow[\varepsilon \searrow 0]{} \varphi^{(m)}$ uniformly on $\mathbb{R}$. Because $(\rho_\varepsilon * \varphi)^{(m)} = \rho_\varepsilon * \varphi^{(m)}$ we have shown that $\rho_\varepsilon * \varphi \xrightarrow[\varepsilon \searrow 0]{} \varphi$ in $\mathcal{D}(\mathbb{R})$ and consequently by $\mathcal{D}(\mathbb{R})$-continuity of u that $\langle \rho_\varepsilon * u, \varphi \rangle \longrightarrow \langle u, \varphi \rangle$.

<u>(iv)</u> Take $\chi = \rho_{\frac{1}{2}} * \mathbb{1}_{(-\frac{3}{2}, \frac{3}{2})}$. Then $\chi \in \mathcal{D}(\mathbb{R})$ and $\mathbb{1}_{(-1,1)} \leq \chi \leq \mathbb{1}_{(-2,2)}$. Define

$$u_j(x) = \chi\left(\tfrac{x}{j}\right)(\rho_{\frac{1}{j}} * u)(x), \quad x \in \mathbb{R}.$$

Then $u_j \in C^\infty(\mathbb{R})$ and $\operatorname{supp}(u_j) \subseteq [-2j, 2j]$ so $u_j \in \mathcal{D}(\mathbb{R})$. If $\varphi \in \mathcal{D}(\mathbb{R})$, say with $\operatorname{supp}(\varphi) \subset [-a,a]$, then for $j > a$

$$\langle u_j, \varphi \rangle = \langle \rho_{\frac{1}{j}} * u, \chi(\tfrac{\cdot}{j})\varphi \rangle = \langle \rho_{\frac{1}{j}} * u, \varphi \rangle$$

and consequently $u_j \to u$ in $\mathcal{D}'(\mathbb{R})$.

(5) Recall that we can characterize the support, $\mathrm{supp}(u)$, of $u \in \mathcal{D}'(\Omega)$ as the smallest relatively closed subset A of Ω so

$$u|_{\Omega \setminus A} = 0.$$

Take $\varphi \in \mathcal{D}(\Omega)$ so $\mathrm{supp}(\varphi) \subset \Omega \setminus \mathrm{supp}(u)$.
Then for a multi-index $\alpha \in \mathbb{N}_0^n$ we have $\mathrm{supp}(\partial^\alpha \varphi) \subseteq \mathrm{supp}(\varphi)$, hence $\langle \partial^\alpha u, \varphi \rangle = (-1)^{|\alpha|} \langle u, \partial^\alpha \varphi \rangle = 0$, so that $\partial^\alpha u$ vanishes on the open set $\Omega \setminus \mathrm{supp}(u)$ and therefore $\mathrm{supp}(\partial^\alpha u) \subseteq \mathrm{supp}(u)$.
Consequently we also have that

$$\langle p(\partial) u, \varphi \rangle = \sum_{|\alpha| \leq k} c_\alpha (-1)^{|\alpha|} \langle u, \partial^\alpha \varphi \rangle$$

$$= 0,$$

so $\mathrm{supp}(p(\partial)u) \subseteq \mathrm{supp}(u)$ holds.

<u>BACKGROUND:</u> Look up Peetre's characterization of partial differential operators.

$\boxed{\text{EX}}$ If H is Heaviside's function, then $\operatorname{supp}(H) = [0,\infty)$, but $H' = \delta_0$ has $\operatorname{supp}(H') = \{0\}$.

Let $u \in \mathcal{D}'(\Omega)$ and recall that the singular support, $\operatorname{sing.supp}(u)$, can be characterized as the smallest relatively closed subset A of Ω so $u|_{\Omega \setminus A} \in C^\infty(\Omega \setminus A)$.

If $\Omega \setminus \operatorname{sing.supp}(u) \neq \emptyset$, then as distributional derivatives coincide with the usual derivatives for C^∞ functions we get for $\varphi \in \mathcal{D}(\Omega)$ with $\operatorname{supp}(\varphi) \subset \Omega \setminus \operatorname{sing.supp}(u)$:

$$\langle p(\partial)u, \varphi \rangle = \langle u, \sum_{|\alpha| \leq k} c_\alpha (-1)^{|\alpha|} \partial_\alpha \varphi \rangle$$

$$= \langle p(\partial)(u|_{\Omega \setminus \operatorname{sing.supp}(u)}), \varphi \rangle$$

Consequently, $(p(\partial)u)|_{\Omega \setminus \operatorname{sing.supp}(u)} =$

$p(\partial)(u|_{\Omega \setminus \operatorname{sing.supp}(u)}) \in C^\infty(\Omega \setminus \operatorname{sing.supp}(u))$

and therefore $\operatorname{sing.supp}(p(\partial)u) \subseteq \operatorname{sing.supp}(u)$.

$\boxed{\text{Remark}}$ $p(\partial)$ is hypoelliptic if for all $u \in \mathcal{D}'(\Omega)$, $\operatorname{sing.supp}(p(\partial)u) = \operatorname{sing.supp}(u)$ holds.

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$\boxed{\text{EX}}$ Let $\{q_j : j \in \mathbb{N}\}$ be an enumeration of the rationals and define

$$u(x,y) = \sum_{j=1}^{\infty} 2^{-j} \mathbb{1}_{(-\infty, q_j] \times \mathbb{R}}(x,y), \quad (x,y) \in \mathbb{R}^2.$$

Then $u(x,y)$ is constant with respect to $y \in \mathbb{R}$ and an decreasing function with respect to $x \in \mathbb{R}$, so u is a regu= lar distribution. Because $\mathbb{Q}$ is dense we have $\text{sing supp}(u) = \mathbb{R}^2$. But clearly $\partial_y u = 0$ in $\mathcal{D}'(\mathbb{R}^2)$ so $\text{sing supp}(\partial_y u) = \emptyset$.

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6 Optional

First we note that the set of zeros $Z := \{z_j : j \in J\}$ cannot have any limit points in $\mathbb{C}$ since otherwise $F \equiv 0$ by the identity theorem. Consequently we have for each $w \in \mathbb{C}$ that for sufficiently small $r > 0$ the disc $B_r(w)$ contains at most one zero.

If $B_r(w) \cap Z = \emptyset$, then f is continuous on $B_r(w)$ and so locally integrable there. If $B_r(w) \cap Z = \{z_j\}$, then from Taylor's theorem applied to F about z_j we get

$$F(z) = (z - z_j)^{m_j} G(z)$$

where $G : \mathbb{C} \to \mathbb{C}$ is entire and without zeros in $B_r(w)$. By properties of the logarithm

$$f = m_j \log|\cdot - z_j| + \log|G| \quad \text{on } B_r(w).$$

The first term is in $L^1(B_r(w))$ (by checking in polar coordinates about z_j) and the second term is continuous on $B_r(w)$, thus $f \in L^1_{loc}(B_r(w))$. But then $f \in L^1_{loc}(\mathbb{C})$ and f is a regular distribution on $\mathbb{C}$. By localization properties of distributions (more precisely Theorem 6.1 in LN) it suffices to consider $f|_{B_r(w)}$ in the two cases above: $\mathrm{card}(B_r(w) \cap Z) \in \{0,1\}$.

If $B_r(w) \cap Z = \emptyset$ we calculate with $\Delta = 4 \frac{\partial^2}{\partial \bar{z} \partial z}$ on $B_r(w)$:

$$\Delta f = 2\frac{\partial}{\partial \bar{z}}\left(\frac{\partial}{\partial z}\log\left(F(z)\overline{F(z)}\right)\right)$$

$$= 2\frac{\partial}{\partial \bar{z}}\left(\frac{1}{F(z)\overline{F(z)}}\left(\frac{\partial}{\partial z}F(z)\cdot\overline{F(z)} + F(z)\frac{\partial}{\partial z}\overline{F(z)}\right)\right)$$

$$= 2\frac{\partial}{\partial \bar{z}}\left(\frac{F'(z)}{F(z)} + 0\right)$$

$= 0$ by the Cauchy-Riemann equations applied twice.

If $B_r(w) \cap Z = \{z_j\}$, then on $B_r(w)$:

$$\Delta f = \Delta\left(m_j \log|\cdot - z_j| + \log|G|\right)$$

$$= m_j \delta_{z_j}$$

by above and EX 4.22 in LN.