B4.3 Distribution Theory MT20

Lecture 1:
- Why distributions?
- Start on test functions.

Present definition and theory initiated by Laurent Schwartz 1940-50

- Why distributions?

Consider PDES

① $u_{xy} = 0$ ② $u_{yx} = 0$

① and ② equivalent if $u = u(x,y)$ twice continuously differentiable, but not in general!

$u = h(x)$ solves ① for <u>all</u> h, but not ② when h not differentiable.

Awkward situation — need more flexibility and it would be nice if we always could differentiate. So need objects that can represent derivatives of nondifferentiable functions!

= distributions.

To motivate definition consider inhomogeneous PDE

$$\boxed{u_{xy} = f \quad \text{in } \mathbb{R}^2}$$

$f : \mathbb{R}^2 \to \mathbb{R}$ given continuous function.

If u twice continuously differentiable and $\varphi : \mathbb{R}^2 \to \mathbb{R}$ twice continuously differentiable and vanishing off a bounded set, then

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f\varphi \, dx\, dy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u_{xy}\varphi \, dx\, dy$$

$$\overset{\text{parts}}{=} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u\,\varphi_{yx} \, dx\, dy \quad (*)$$

$$\boxed{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u\varphi_{yx}\,dx\,dy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f\varphi\,dx\,dy} \quad (*)$$

(*) holds then for <u>all</u> such φ.

Note if (*) holds and u is twice cont. diff., then $u_{xy} = f$ too.

But (*) makes sense for all continuous u! Declare u solves PDE $u_{xy} = f$ in <u>weak sense</u> if (*) holds.

Then $u_{xy} = f$ and $u_{yx} = f$ are equivalent in weak sense because

$$\varphi_{yx} = \varphi_{xy}$$

when φ twice cont. diff.

Distribution theory goes further:

it defines u_{xy} to be linear functional

$$\varphi \longmapsto \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u\,\varphi_{xy}\,dx\,dy$$

also when no continuous f exists as in (*). Distribution theory is the theory of such linear functionals.

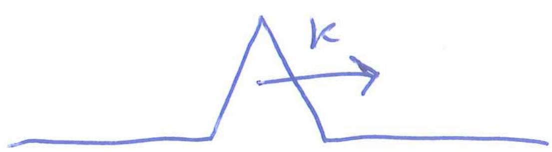
But isn't there an easier way to do it?

1-D Wave Equation $u_{tt} = k^2 u_{xx}$

- Models vibrating string

$u = f(x - kt)$ travelling wave of shape $y = f(x)$ moving to the right with speed k.

Solves WE when f twice diff., but makes physical sense also if f isn't diff., eg



Dismiss physically sound solution because of technicality? ⇝ No.
Possible remedy: accept as solution any function that satisfies PDE at all but, say, finitely many points.

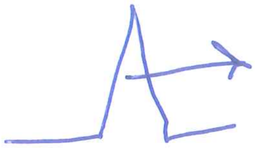
Too simplistic, doesn't work!

2-D Laplace Equation $\Delta u = u_{xx} + u_{yy} = 0$

- Models electric potential in region with no external charges. Experience tells us such potentials are smooth.

But $u = G_0(x,y) = \frac{1}{4\pi} \log(x^2+y^2)$ is a solution in $\mathbb{R}^2 \setminus \{(0,0)\}$. It can't be extended in a smooth manner to $\mathbb{R}^2$, so can't be a solution there!

Distribution theory can distinguish between the two situations:

 solves 1-D WE in sense of distributions,

$\Delta G_0 = \delta_0$ in sense of distributions.

Dirac's delta function, a distribution.

Another example

$f \in L^p(\mathbb{R}) \qquad 1 \leq p \leq \infty$

- f not really a function $\mathbb{R} \to \mathbb{R}$...

$A \subset \mathbb{R}$ measurable, define

$$\langle f, \mathbb{1}_A \rangle := \int_A f \, dx$$

Well-defined: doesn't depend on representative chosen to calculate the integral.

- Knowing $\langle f, \mathbb{1}_A \rangle$ for all A determines f uniquely as L^p function offers us an alternative view of f.

We can extend bracket:

$$\langle f, \phi \rangle := \int_{\mathbb{R}} f \phi \, dx$$

to $\phi \in L^{\text{simple}}(\mathbb{R})$

or to $\phi \in L^q(\mathbb{R})$

$\frac{1}{p} + \frac{1}{q} = 1$

- $\mathbb{1}_A$ or ϕ acts as a 'measurement' of f, or as a 'test function'

Test functions

We start with some definitions and notation.

$\emptyset \neq \Omega \subseteq \mathbb{R}^n$ open set

- $f : \Omega \to \mathbb{R}$ <u>continuously differentiable</u> if f is continuous and $\partial_1 f, \dots, \partial_n f$ exist and are continuous in Ω.

Note: $x \longmapsto \partial_1 f(x)$ must be cont. jointly in $x = (x_1, \dots, x_n)$

Notation: $\partial_1 f(x) := \dfrac{\partial f}{\partial x_1}(x)$ etc.

Generalize to $k \in \mathbb{N}$ derivatives:

- f <u>k times continuously differentiable</u>

if f is continuous and all partial derivatives up to and including order k exist and are continuous in Ω.

Notation

$$C(\Omega) = C^0(\Omega) := \{ f : \Omega \to \mathbb{R} \text{ cont.} \}$$

$$C^k(\Omega) := \{ f : \Omega \to \mathbb{R} \ \ k \text{ times cont. diff.} \}$$

$$C^\infty(\Omega) := \bigcap_{k=1}^{\infty} C^k(\Omega) \quad \text{infinitely often diff. functions}$$

If $f \in C^k(\Omega)$, then we call f a C^k function etc.

$\boxed{\text{Lemma}}$ If $f \in C^2(\mathbb{R}^n)$, then

$$\partial_j \partial_k f = \partial_k \partial_j f.$$

$\boxed{\text{Pf.}}$ Denote $\Delta_h f(x) := f(x+h) - f(x)$,

$(e_i)_{i=1}^n$ standard basis for $\mathbb{R}^n$.

For $s, t \in \mathbb{R} \setminus \{0\}$: check that

$$\Delta_{se_j} \Delta_{te_k} f(x) = \Delta_{te_k} \Delta_{se_j} f(x)$$

FTC twice

$$= st \int_0^1 \int_0^1 (\partial_k \partial_j f)(x + s\rho e_j + t\sigma e_k) \, d\rho \, d\sigma$$

Thus $\dfrac{\Delta_{se_j} \Delta_{te_k} f(x)}{st} = \dfrac{\Delta_{te_k} \Delta_{se_j} f(x)}{st}$

$$= \int_0^1 \int_0^1 (\partial_k \partial_j f)(x + s\rho e_j + t\sigma e_k) \, d\rho \, d\sigma$$

$\longrightarrow (\partial_k \partial_j f)(x)$ as $s, t \longrightarrow 0$. $\square$

Order in which we partially diff. doesn't matter for C^k functions.

$\Rightarrow$ Justifies use of <u>multi-index notation for partial derivatives</u>!

An n-dimensional multi-index or simply a <u>multi-index</u>:

$$\alpha \in \mathbb{N}_0^n \quad , \quad \alpha = (\alpha_1, \dots, \alpha_n)$$

Length (or order) of α : $|\alpha| := \alpha_1 + \dots + \alpha_n$

Note: If $\alpha, \beta \in \mathbb{N}_0^n$ and $t \in \mathbb{N}_0$, then
$\alpha + t\beta := (\alpha_1 + t\beta_1, \dots, \alpha_n + t\beta_n) \in \mathbb{N}_0^n$,

$$\alpha! := \alpha_1! \cdots \alpha_n! \qquad (0! := 1)$$

$$x^\alpha := x_1^{\alpha_1} \cdot \ldots \cdot x_n^{\alpha_n} \qquad (x^0 := 1)$$

Assume $f \in C^k(\Omega)$, $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \leq k$

Then

$$\partial^\alpha f := \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}.$$

Note: if $\alpha, \beta \in \mathbb{N}_0^n$ and $|\alpha| + |\beta| \leq k$,

then

$$\partial^{\alpha+\beta} f = \partial^\alpha \partial^\beta f = \partial^\beta \partial^\alpha f$$

(and $\partial^0 f := f$).

In this notation formulas have the same form in n-dimensions as in 1-dimension:

Taylor's formula

open ball with centre x_0 and radius r

$$f \in C^k(B_r(x_0))$$

For $x \in B_r(x_0)$

$$f(x) = \sum_{|\alpha|<k} \frac{\partial^\alpha f(x_0)}{\alpha!}(x-x_0)^\alpha + k\int_0^1 (1-t)^{k-1} \sum_{|\alpha|=k} \frac{\partial^\alpha f(x_0+t(x-x_0))}{\alpha!}(x-x_0)^\alpha \, dt$$

(Pf: apply 1-D Taylor to

$$s \mapsto f(x_0 + s(x-x_0))$$

at $s=0$.)

Generalized Leibniz rule

If $f, g \in C^k(\Omega)$, then $fg \in C^k(\Omega)$

and for $|\alpha| \leq k$

$$\partial^\alpha (fg) = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \partial^{\alpha-\beta} f \, \partial^\beta g$$

where $\beta \leq \alpha$ means

$\beta_1 \leq \alpha_1 , \ \dots , \ \beta_n \leq \alpha_n$

and

$$\binom{\alpha}{\beta} := \binom{\alpha_1}{\beta_1} \cdot \ldots \cdot \binom{\alpha_n}{\beta_n} .$$

(Pf: by induction on $|\alpha|$ and use of usual Leibniz rule in induction step.)

Support of cont. function

$f \in C(\Omega)$

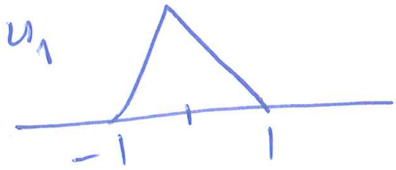
$\operatorname{supp}(f) := \Omega \cap \overline{\{f \neq 0\}}$

so it's the closure of $\{f \neq 0\}$ relative to Ω.

<u>Note</u> $\operatorname{supp}(f)$ is closed relative to Ω, but not necessarily in $\mathbb{R}^n$.

$\boxed{\text{EX}}$ $u_1(x) = \max\{0, 1-|x|\}$, $x \in \mathbb{R}$

is cont. and $\operatorname{supp}(u_1) = [-1, 1]$



$u_2(x) = \max\{0, 1-|x|\}$, $x \in (-1, 1)$.

is cont. and $\operatorname{supp}(u_2) = (-1, 1)$

$\Longrightarrow$ Support depends on Ω.

Compact support & test functions

Let $k \in \mathbb{N}_0 \cup \{\infty\}$.

$$C_c^k(\Omega) := \left\{ u \in C^k(\Omega) : \begin{array}{l} \text{supp}(u) \text{ is} \\ \text{compact} \end{array} \right\}$$

Note: $\emptyset$ is compact, so $0 \in C_c^k(\Omega)$.

and if $u \in C_c^k(\Omega)$, then

$$\text{dist}(\text{supp}(u), \partial\Omega) > 0$$

$$\text{dist}(\text{supp}(u), \partial\Omega) = \inf\left\{ |x-y| : \begin{array}{l} x \in \text{supp}(u) \\ y \in \partial\Omega \end{array} \right\}.$$

(By definition $\text{dist}(\emptyset, \partial\Omega) := \infty$)

$\varphi : \Omega \to \mathbb{R}$ (or into $\mathbb{C}$ ...) is a <u>test function</u> if $\varphi \in C_c^\infty(\Omega)$

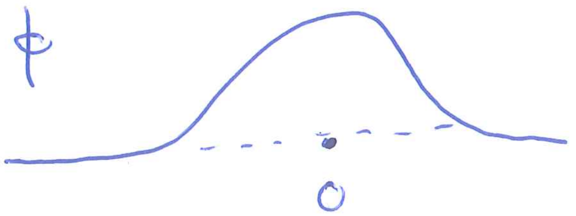
<u>Notation</u> $\mathcal{D}(\Omega) := C_c^\infty(\Omega)$

$\boxed{\text{EX}}$ $f(x) := \begin{cases} e^{-\frac{1}{x}}, & x > 0 \\ 0, & x \leq 0. \end{cases}$

On sheet 1 show $f \in C^\infty(\mathbb{R})$.

Put $\phi(x) := f(1-|x|^2)$, $x \in \mathbb{R}^n$.

Then $\phi \in C^\infty(\mathbb{R}^n)$ by Chain rule and $\text{supp}(\phi) = \overline{B_1(0)}$ (the closed unit ball)



a bump function

Can we make bump on $\overline{B_r(x_0)}$?

— Yes, take

$$\varphi(x) := \phi\left(\frac{x - x_0}{r}\right), \quad x \in \mathbb{R}^n.$$