Lecture 7:

- Exx of calculation of distributional derivatives
- Consistency of distributional partial derivatives for C^1 functions
- Mollification and approximation

(pp. 36-40 in lecture notes)

EX Heaviside's function is

$$H(x) = \begin{cases} 0, & x<0 \\ 1, & x \geq 0 \end{cases} \quad \left(= \mathbb{1}_{[0,\infty)}(x)\right)$$

Clearly $H \in L^1_{loc}(\mathbb{R})$ and $H' = \delta_0$ since

$$\langle H', \phi \rangle = \langle H, -\phi' \rangle = -\int_0^\infty \phi' \, dx = \phi(0).$$

$\boxed{\text{EX}}$ Calculate the distributional derivatives of

(i) $\cos x$ (ii) $\cos x + H(x)$

(iii) $\sin x \, H(x)$ (iv) $|x|$

(v) a C^1 function $f : \mathbb{R}\setminus\{0\} \to \mathbb{C}$

so $f' \in L^1_{loc}(\mathbb{R})$.

(vi) $\dfrac{1}{|x|^\alpha}$ when $\alpha \in (0,1)$.

<u>(i):</u> cos is C^1, so $\cos' = -\sin$

by consistency (so we can under= stand $(\cdot)'$ both in usual and distri= butional sense)

<u>(ii):</u> Differentiation is a linear operation so

$$(\cos + H)' = \cos' + H' = -\sin + \delta_0$$

$\sin x\, H(x)$

(iii): We use Leibniz' rule (allowed since $\sin \in C^\infty(\mathbb{R})$, $H \in \mathcal{D}'(\mathbb{R})$):

$$(\sin x H)' = \cos x H + \sin x \delta_0$$

$\uparrow$ can be simplified!

$$\langle \sin x\, \delta_0, \phi \rangle = \langle \delta_0, \sin x\, \phi \rangle = 0$$

so $(\sin x H)' = \cos x\, H$

(iv): $|x|$ isn't diff. at 0, but is piecewise C^1. For $\phi \in \mathcal{D}(\mathbb{R})$:

$$\left\langle \frac{d}{dx}|x|, \phi \right\rangle = -\int_{-\infty}^{\infty} |x| \phi'(x)\, dx =$$

$$-\int_0^\infty x\phi'(x)\, dx + \int_{-\infty}^0 x\phi'(x)\, dx \overset{\text{parts}}{=}$$

$$-\Big[x\phi(x)\Big]_{x\to 0+}^{x\to\infty} + \int_0^\infty \phi(x)\, dx$$

$$+\Big[x\phi(x)\Big]_{x\to-\infty}^{x\to 0-} - \int_{-\infty}^0 \phi(x)\, dx = \int_{-\infty}^{\infty} (\mathbb{1}_{(0,\infty)} - \mathbb{1}_{(-\infty,0)})\phi\, dx$$

so $\quad \frac{d}{dx}|x| = \mathbb{1}_{(0,\infty)} - \mathbb{1}_{(-\infty,0)}$

it doesn't matter if intervals are open or closed here – why?

<u>(v):</u> $f: \mathbb{R}\setminus\{0\} \to \mathbb{C}$ is C^1 and $f' \in L^1_{loc}(\mathbb{R})$. Find distributional derivative (so T_f' ...)

But is f determining a distribution at all? Since nothing else is mentioned we expect that the assumptions imply that $f \in L^1_{loc}(\mathbb{R})$. We check it!

Define (arbitrarily) $f(0) = 0$.
Then f is measurable and if $0 < y < x$, then we get from

FTC (applicable since f is C^1 on $\mathbb{R}^+$):

$$f(x) - f(y) = \int_y^x f'(t)\,dt .$$

Because $f' \in L^1_{loc}(\mathbb{R})$ we have

$$f(x) - f(y) = \int_y^x f'(t)\,dt \to 0 \text{ as } x, y \to 0+$$

and so $(f(x))_{x>0}$ is a Cauchy family in $\mathbb{C}$ as $x \to 0+$, hence

$$f(0+) = \lim_{x \to 0+} f(x) \quad \text{exists in } \mathbb{C}.$$

Likewise $f(0-) = \lim_{x \to 0-} f(x)$ exists in $\mathbb{C}$, so f is piecewise continuous. In particular, $f \in L^1_{loc}(\mathbb{R})$.

Now for $\phi \in \mathcal{D}(\mathbb{R})$ calculate:

$$\langle f', \phi \rangle = -\langle f, \phi' \rangle = -\int_{-\infty}^{\infty} f \phi'\,dx =$$

$$-\int_0^\infty f\phi'\,dx - \int_{-\infty}^0 f\phi'\,dx \overset{\text{parts}}{=}$$

$$-\left[f\phi\right]_{x\to 0+}^{x\to\infty} + \int_0^\infty f'\phi\,dx$$

$$-\left[f\phi\right]_{x\to -\infty}^{x\to 0-} + \int_{-\infty}^0 f'\phi\,dx =$$

$$f(0+)\phi(0) + \int_0^\infty f'\phi\,dx - f(0-)\phi(0) + \int_{-\infty}^0 f'\phi\,dx$$

$$= \big(f(0+) - f(0-)\big)\phi(0) + \int_{-\infty}^\infty f'\phi\,dx$$

$$T_f' = \underset{\substack{\uparrow \\ \text{usual derivative}}}{f'} + \big(f(0+) - f(0-)\big)\delta_0$$

<u>(vi):</u> $\frac{1}{|x|^\alpha}$, $0 < \alpha < 1$

$\frac{1}{|x|^\alpha} \in L^1_{loc}(\mathbb{R})$ and C^1 on $\mathbb{R}\setminus\{0\}$

with usual derivative

$$-\alpha \frac{1}{|x|^{\alpha+2}}x, \quad x \neq 0.$$

Not locally integrable: problem at 0.

But any distribution has a distri= butional derivative!

Let $\phi \in \mathcal{D}(\mathbb{R})$ and calculate:

$$< \frac{d}{dx} |x|^{-\alpha}, \phi > = -\int_{-\infty}^{\infty} \frac{\phi'(x)}{|x|^{\alpha}}\, dx =$$

$$-\left(\int_{-\infty}^{0} + \int_{0}^{\infty} \right) \frac{\phi'(x)}{|x|^{\alpha}}\, dx$$

Would like to integrate by parts on $(-\infty, 0)$ and $(0, \infty)$ but we must accommodate $|x|^{-\alpha-1} \notin L^1_{loc}$!

Note $(\phi(x) - \phi(0))' = \phi'(x)$ so

$$\int_{0}^{\infty} \frac{\phi'(x)}{|x|^{\alpha}}\, dx = \lim_{\varepsilon \to 0+} \left\{ \left[\frac{\phi(x) - \phi(0)}{|x|^{\alpha}} \right]_{x=\varepsilon}^{x = 1/\varepsilon} \right.$$

$$\left. + \int_{\varepsilon}^{1/\varepsilon} \alpha x^{-\alpha-1} (\phi(x) - \phi(0))\, dx \right\}$$

Since $\frac{\phi(x) - \phi(0)}{x} \to \phi'(0)$ as $x \to 0$

we have (since $0<\alpha<1$)

$$\left[\frac{\phi(x)-\phi(0)}{x^{\alpha}} \right]_{x=\varepsilon}^{x=\frac{1}{\varepsilon}} \to 0 \quad \text{as } \varepsilon \to 0+$$

and

$$\frac{\phi(x)-\phi(0)}{|x|^{\alpha+1}} \in L^1(\mathbb{R}).$$

Proceeding similarly for $x<0$ we find

$$\left\langle \frac{d}{dx}|x|^{-\alpha}, \phi \right\rangle = -\alpha \int_{-\infty}^{\infty} \frac{\phi(x)-\phi(0)}{|x|^{\alpha+1}} \frac{x}{|x|} dx$$

The distribution

$$\phi \longmapsto \int_{-\infty}^{\infty} \frac{\phi(x)-\phi(0)}{|x|^{\alpha+1}} \frac{x}{|x|} dx$$

is written

$$\text{fp.}\left(\frac{x}{|x|^{\alpha+2}} \right) \qquad \boxed{0<\alpha<1}$$

and called 'finite part of $\frac{x}{|x|^{\alpha+2}}$'.

For calculating higher order distributional derivatives of $|x|^{-\alpha}$ one can express the resulting distributions by subtracting suitable Taylor polynomials from ϕ. We shall return to this on problem sheets.

Consistency of distributional and usual partial derivatives for C^1 functions

$\emptyset \neq \Omega \subseteq \mathbb{R}^n$ open subset

Let $f : \Omega \to \mathbb{C}$ be C^1 and $j \in \{1, \dots, n\}$

Claim: $\partial_j f = \dfrac{\partial f}{\partial x_j}$

distributional   usual

Step 1 Let $\phi \in \mathcal{D}(\Omega)$ and assume

$\operatorname{supp} \phi \subseteq R = (a_1, b_1) \times \ldots \times (a_n, b_n)$

where $\overline{R} \subset \Omega$.

Then $\langle \partial_j f, \phi \rangle = \langle f, -\frac{\partial \phi}{\partial x_j} \rangle =$

$$-\int_R f \frac{\partial \phi}{\partial x_j} \, dx$$

Write $x = (x_j, x') \in R = (a_j, b_j) \times R'$.

Since $f \frac{\partial \phi}{\partial x_j} \in L^1(R)$ we can use

Fubini :

$$\int_R f \frac{\partial \phi}{\partial x_j} \, dx = \int_{R'} \int_{a_j}^{b_j} f(x_j, x') \frac{\partial \phi}{\partial x_j}(x_j, x') \, dx_j \, dx'$$

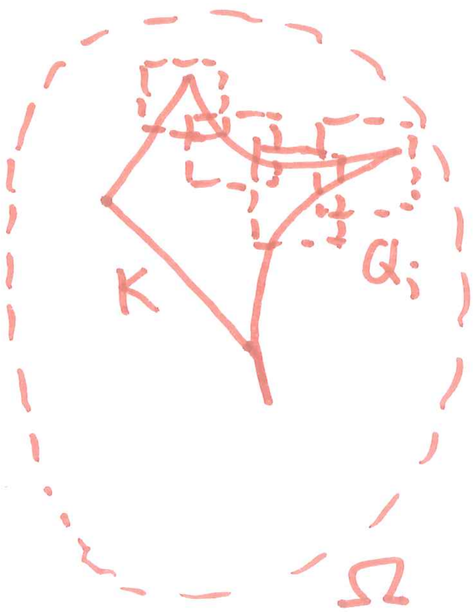
parts

$$= \int_{R'} \left\{ \left[f(x_j, x') \phi(x_j, x') \right]_{x_j = a_j}^{x_j = b_j} - \int_{a_j}^{b_j} \frac{\partial f}{\partial x_j} \phi \, dx_j \right\} dx'$$

$$= -\int_R \frac{\partial f}{\partial x_j} \phi \, dx = -\int_\Omega \frac{\partial f}{\partial x_j} \phi \, dx .$$

<u>Step 2</u> Let $\phi \in \mathcal{D}(\Omega)$.

$K = \operatorname{supp} \phi$ compact subset of Ω.



Cover K by finitely many small open cubes Q_i, $i \in I$, so $\overline{Q_i} \subset \Omega$:

$$K \subset \bigcup_{i \in I} Q_i , \quad \overline{Q_i} \subset \Omega$$

I finite index set

We now use <u>Theorem 2.3</u> from lecture notes: find a smooth partition of unity for K that is subordinated Q_i, $i \in I$:

$\phi_i \in \mathcal{D}(\Omega)$, $0 \leqq \phi_i \leqq 1$,

$\operatorname{supp} \phi_i \subseteq Q_i$, $\sum_{i \in I} \phi_i \leqq 1$ on Ω,

$\sum_{i \in I} \phi_i = 1$ on K.

Now $\phi = \sum_{i \in I} \phi \phi_i$ and

$\phi \phi_i \in \mathcal{D}(\Omega)$, $\operatorname{supp}(\phi \phi_i) \subseteq \overline{Q}_i \subset \Omega$

so by Step 1

$$\langle \partial_j f, \phi \phi_i \rangle = -\int_\Omega \frac{\partial f}{\partial x_j} \phi \phi_i \, dx .$$

Sum over $i \in I$ to get

$$\langle \partial_j f, \phi \rangle = -\int_\Omega \frac{\partial f}{\partial x_j} \phi \, dx .$$

Mollification & approximation

Recall that we defined 'convolution with test function' for distributions:

If $u \in \mathcal{D}'(\mathbb{R}^n)$, $\theta \in \mathcal{D}(\mathbb{R}^n)$, then $\theta * u$ $(= u * \theta) \in \mathcal{D}'(\mathbb{R}^n)$ is defined by

$$\langle \theta * u, \phi \rangle := \langle u, \tilde{\theta} * \phi \rangle, \phi \in \mathcal{D}(\mathbb{R}^n).$$

Here $\tilde{\theta}(x) := \theta(-x)$.

We also defined 'differentiation' and writing out definitions we easily find for a multi-index $\alpha \in \mathbb{N}_0^n$:

$$\partial^\alpha(\theta * u) = (\partial^\alpha \theta) * u = \theta * (\partial^\alpha u).$$

Here we're interested in convolving with ρ_ε from the standard mollifier $(\rho_\varepsilon)_{\varepsilon>0}$ on $\mathbb{R}^n$:

$$\rho_\varepsilon * u = u * \rho_\varepsilon \in \mathcal{D}'(\mathbb{R}^n),$$

$$\langle \rho_\varepsilon * u, \phi \rangle = \langle u, \rho_\varepsilon * \phi \rangle, \phi \in \mathcal{D}(\mathbb{R}^n)$$

because ρ_ε in particular is even so $\tilde{\rho}_\varepsilon = \rho_\varepsilon$.

$\boxed{\text{Lemma}}$ (= lemma 4.12 in lecture notes)

For each $\varepsilon > 0$, $\rho_\varepsilon * u \in C^\infty(\mathbb{R}^n)$

and

$$(\rho_\varepsilon * u)(x) = \langle u, \rho_\varepsilon(x - \cdot) \rangle,$$

that is, u acting on test function

$$y \longmapsto \rho_\varepsilon(x-y).$$

Furthermore,

$$\rho_\varepsilon * u \longrightarrow u \quad \text{in} \quad \mathcal{D}'(\mathbb{R}^n) \text{ as } \varepsilon \to 0$$

The first part of the proof relies on the following

$\boxed{\text{Auxiliary Lemma}}$ If $u \in \mathcal{D}'(\mathbb{R}^n)$, $\theta \in \mathcal{D}(\mathbb{R}^n)$ and $h(x) := \langle u, \theta(x - \cdot) \rangle$, $x \in \mathbb{R}^n$, then $h \in C^1(\mathbb{R}^n)$ and $\partial_j h(x) = \langle u, (\partial_j \theta)(x - \cdot) \rangle$.

<u>Sketch of proof.</u> Let $(e_j)_{j=1}^n$ be the standard basis for $\mathbb{R}^n$. Consider for $t \in \mathbb{R} \setminus \{0\}$:

$$\frac{h(x+te_j) - h(x)}{t} = \left\langle u, \frac{\theta(x+te_j - \cdot) - \theta(x - \cdot)}{t} \right\rangle$$

Here, for fixed x and j;

$$\frac{\theta(x+te_j-y)-\theta(x-y)}{t} \to (\partial_j\theta)(x-y)$$

uniformly in $y \in \mathbb{R}^n$ as $t \to 0$.

(Inspection shows that in fact the convergence is uniform in $x \in \mathbb{R}^n$ too.)

The support of $y \mapsto \theta(x-y)$ is $x - \operatorname{supp}\theta = \{x-y : y \in \operatorname{supp}\theta\}$

so the support of

$$y \mapsto \frac{\theta(x+te_j-y)-\theta(x-y)}{t}$$

is for $0<|t|<1$ contained in

$$\{x-y : y \in \overline{B_1(\operatorname{supp}\theta)}\}$$

a <u>fixed compact</u> set in $\mathbb{R}^n$.

Next, for any $\alpha \in \mathbb{N}_0^n$,

$$\frac{\partial_y^\alpha \theta(x+te_j-y) - \partial_y^\alpha \theta(x-y)}{t} \longrightarrow \partial_y^\alpha (\partial_j \theta)(x-y)$$

uniformly in $y \in \mathbb{R}^n$ as $t \to 0$.

(Again, inspection shows that in fact the convergence is uniform in $x \in \mathbb{R}^n$ too.)

Consequently we have shown that

$$\frac{\theta(x+te_j-\cdot) - \theta(x-\cdot)}{t} \to (\partial_j \theta)(x-\cdot)$$

in $\mathcal{D}(\mathbb{R}^n)$ as $t \to 0$, hence by $\mathcal{D}$-continuity of u that

$$\frac{h(x+te_j) - h(x)}{t} \xrightarrow[t \to 0]{} \langle u, \partial_j \theta(x-\cdot)\rangle$$

Because the map

$$\mathbb{R}^n \ni x \longmapsto \partial_j \theta(x - \cdot) \in \mathcal{D}(\mathbb{R}^n)$$

is continuous in the sense

$$\partial_j \theta(x - \cdot) \to \partial_j \theta(x_0 - \cdot) \text{ in } \mathcal{D}(\mathbb{R}^n)$$

as $x \to x_0$ we also have that

$$x \longmapsto \partial_j h(x) = \langle u, \partial_j \theta(x - \cdot) \rangle$$

is continuous. □

An induction argument on the length of the multi-index $\alpha \in \mathbb{N}_0^n$ now shows

$$x \longmapsto \langle u, \rho_\varepsilon(x - \cdot) \rangle$$

is a C^∞ function with α^{th} derivative

$$x \longmapsto \langle u, \partial_x^\alpha \rho_\varepsilon(x - \cdot) \rangle .$$

Remember we must show

$$\rho_\varepsilon * u \in C^\infty(\mathbb{R}^n),$$

$$(\rho_\varepsilon * u)(x) = \langle u, \rho_\varepsilon(x - \cdot)\rangle$$

we know this a C^∞ function

$$\langle \rho_\varepsilon * u, \phi \rangle = \langle u, \rho_\varepsilon * \phi \rangle$$

$$(\rho_\varepsilon * \phi)(x) = \int_{\mathbb{R}^n} \rho_\varepsilon(x-y)\phi(y)\,dy$$

smooth compactly supported

The integral can be understood as a Riemann integral so is the limit of Riemann sums.

Notation Dyadic cubes in $\mathbb{R}^n$

For $k \in \mathbb{N}$, a k-th generation dyadic cube is a cube Q of form

$$Q = c^Q + (0, 2^{-k}]^n$$

$$= (c_1^Q, c_1^Q + 2^{-k}] \times \cdots \times (c_n^Q, c_n^Q + 2^{-k}]$$

where the left corner point c^Q belong to the dilated integer grid $2^{-k}\mathbb{Z}^n$.

Collection of all k-th generation dyadic cubes denoted $\mathcal{D}_k$:

Note $\mathbb{R}^n = \bigcup \mathcal{D}_k$ disjoint union.

Fix $x \in \mathbb{R}^n$, $k \in \mathbb{N}$.

$$R_k(x) = \sum_{Q \in D_k} \rho_\varepsilon(x - c^Q)\phi(c^Q)\mathcal{L}^n(Q)$$

is a Riemann sum for

$$(\rho_\varepsilon * \phi)(x) = \int_{\mathbb{R}^n} \rho_\varepsilon(x-y)\phi(y)\,dy.$$

Note the sum is finite and that the side length of $Q \in D_k$ is 2^{-k}.

Therefore $R_k(x) \xrightarrow[k\to\infty]{} (\rho_\varepsilon * \phi)(x)$.

Inspection shows the convergence is uniform in $x \in \mathbb{R}^n$.

But in fact

$$\partial^\alpha R_k(x) = \sum_{Q \in D_k} \partial_x^\alpha \rho_\varepsilon(x - c^Q)\phi(c^Q)\mathcal{L}^n(Q)$$

is a Riemann sum for

$$\partial_x^\alpha(\rho_\varepsilon * \phi)(x) = (\partial^\alpha \rho_\varepsilon * \phi)(x)$$

and so also converges uniformly in $x \in \mathbb{R}^n$ as $k \to \infty$.

Finally note

$$\operatorname{supp} R_k \subseteq \overline{B_\varepsilon(\operatorname{supp}\phi)} \quad \forall k$$

so

$$R_k \to \rho_\varepsilon * \phi \quad \text{in } \mathcal{D}(\mathbb{R}^n) \text{ as } k \to \infty.$$

Then by linearity and $\mathcal{D}$-continuity of u:

$$\langle \rho_\varepsilon * u, \phi \rangle = \langle u, \rho_\varepsilon * \phi \rangle =$$

$$\lim_{k \to \infty} \langle u, R_k \rangle = \lim_{k \to \infty} \sum_{Q \in D_k} \langle u, \rho_\varepsilon(\cdot - c^Q) \rangle \phi(c^Q) \mathcal{L}^n(Q)$$

$$= \int_{\mathbb{R}^n} \langle u, \rho_\varepsilon(\cdot - x) \rangle \phi(x) \, dx$$

and we're done since $\tilde{\rho}_\varepsilon = \rho_\varepsilon$.

Finally — the approximation:

$$\rho_\varepsilon * u \longrightarrow u \quad \text{in } \mathcal{D}'(\mathbb{R}^n) \text{ as } \varepsilon \searrow 0$$

This is easy: from a previous ex we know that

$$\rho_\varepsilon * \phi \xrightarrow[\varepsilon \searrow 0]{} \phi \quad \text{in } \mathcal{D}(\mathbb{R}^n)$$

when $\phi \in \mathcal{D}(\mathbb{R}^n)$. Thus

$$\langle \rho_\varepsilon * u, \phi \rangle = \langle u, \check{\rho}_\varepsilon * \phi \rangle$$

$$\xrightarrow[\varepsilon \searrow 0]{} \langle u, \phi \rangle . \square$$