B4.1 Functional Analysis I MT 2020: Problem Sheet 1

When not specified, the scalar field $\mathbb{F}$ may be assumed to be either $\mathbb{R}$ or $\mathbb{C}$.

The questions on this problem sheet relate to the videos on sections 1.1, 1.2 and 1.3.

1. Let X be a real or complex vector space and assume that $x \mapsto \|x\|_0$ is a *seminorm*, i.e. a function from X to $[0, \infty)$ which satisfies

(N2) $\|\lambda x\|_0 = |\lambda| \|x\|_0$ for all $\lambda \in \mathbb{F}$ and all $x \in X$;

(N3) $\|x + y\|_0 \leqslant \|x\|_0 + \|y\|_0$ for all $x, y \in X$.

Let $X_0 = \{ x \in X \mid \|x\|_0 = 0 \}$. Show that X_0 is a subspace of X. Let X/X_0 be the associated quotient space (as defined in Part A Linear Algebra). "Define" $\|x + X_0\| = \|x\|_0$ for $x \in X$.

(i) Show that $\|\cdot\|$ is a well-defined map from X/X_0 to $[0, \infty)$.

(ii) Show that $\|\cdot\|$ is a norm on X/X_0.

2. (i) Consider $\mathbb{R}^m$ equipped with the p-norms $\|x\|_p = (\sum_{i=1}^m |x_i|^p)^{1/p}$ respectively with $\|x\|_\infty = \sup_{i=1,\dots,m} |x_i|$. Show that, for any $x = (x_1, \dots, x_m) \in \mathbb{R}^m$ and any $1 \leqslant p < \infty$,

$$\|x\|_\infty \leqslant \|x\|_p \leqslant m^{1/p} \|x\|_\infty$$

and deduce that $\lim_{p\to\infty} \|x\|_p = \|x\|_\infty$.

(ii) Consider now the sequence space ℓ^p and prove that for any $1 \leqslant p < q \leqslant \infty$

$$\ell^p \subsetneq \ell^q.$$

Hint: Use that $t^q \leqslant t^p$ for all $t \in [0, 1]$

(iii) Consider now the function spaces $L^p([0, 1])$. Prove that if $1 \leqslant p < q \leqslant \infty$ then

$$L^p([0, 1]) \supset L^q([0, 1]).$$

Hint: Use Hölder's inequality with exponents $r = \frac{q}{p}$ and s so that $\frac{1}{r} + \frac{1}{s} = 1$

Does this relation also hold true for $L^p(\mathbb{R})$? (Hint: Compare with part (ii))

3. (i) Consider the space $\mathcal{F}^b(\mathbb{R})$ of all bounded real-valued functions on $\mathbb{R}$ with the sup norm, which you may assume is a Banach space. Show that

$$C_0(\mathbb{R}) := \{f \in C(\mathbb{R}) : |f(t)| \to 0 \text{ as } |t| \to \infty\}$$

is a closed subspace of $\mathcal{F}^b(\mathbb{R})$ and hence deduce that $(C_0(\mathbb{R}), \|\cdot\|_{sup})$ is a Banach space. *You may use without proof that the uniform limit of a sequence of continuous functions is continuous*

(ii) Prove that

$$(C_c(\mathbb{R}) := \{f \in C(\mathbb{R}) : \text{supp}(f) \text{ is compact}\}, \|\cdot\|_{sup})$$

is not a Banach space.

Hint: Show that $C_c(\mathbb{R})$ is dense in $C_0(\mathbb{R})$ by constructing for every $f \in C_0(\mathbb{R})$ a sequence $f_n \in C_c(\mathbb{R})$ so that $\|f - f_n\|_{sup} \to 0$.

4. We let

$$Z := \{f : [-1, 1] \to \mathbb{R} \text{ Lipschitz continuous }\}$$

and define for $f \in Z$

$$\text{Lip}(f) := \inf\{L \in \mathbb{R} : |f(s) - f(t)| \leqslant L|s - t| \quad \text{for all } s, t \in [-1, 1]\}.$$

We furthermore consider the subspace $X := \{f \in Z : f(0) = 0\}$ of Z.

(i) Show $\|f\|_{\text{Lip}} := \text{Lip}(f)$ defines a norm on X. Does this also define a norm on Z?

(ii) For $f \in X$, show that $\|f\|_{\text{Lip}} \geqslant \|f\|_\infty$. Are the two norms equivalent? Justify your answer with a proof or counterexample.

(iii) Show that $(X, \|\cdot\|_{\text{Lip}})$ is a Banach space.

5. Let $X = c_0$ be the space of sequences (α_j) that converge to zero, equipped with the sup norm. Consider the subsets

$$Y = \{\, (\alpha_j) \colon \alpha_{2j-1} = 0,\ j = 1, 2, \dots \} \text{ and } Z = \{\, (\alpha_j) \mid \alpha_{2j} = j^2 \alpha_{2j-1},\ j = 1, 2, \dots \}.$$

Show that Y and Z are closed subspaces of X and that the element $x = (1, 0, \frac{1}{4}, 0, \frac{1}{9}, 0, \dots)$ lies in the closure of $Y + Z$ but not in $Y + Z$.