Chapter I

Banach spaces

I.1 Definition & basic properties

Let X be a vector space over field $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$

<u>Def. 1</u>: A norm $\|\cdot\|$ is a fn

$$\|\cdot\| : X \to \mathbb{R} \text{ s.t.}$$

$\forall x, y \in X, \quad \lambda \in \mathbb{F}$

(N1) $\|x\| \geq 0$ with "=" iff $x = 0$

(N2) $\|\lambda x\| = |\lambda| \cdot \|x\|$

(N3) $= \Delta \quad \|x+y\| \leq \|x\| + \|y\|$.

The pair $(\underset{\text{v.s.}}{\underset{\uparrow}{X}}, \underset{\text{norm}}{\underset{\uparrow}{\|-\|}})$ is called a normed space.

<u>Remark</u>: If $\|-\|$ is a norm on X then

$$d : X \times X \to \mathbb{R}$$
$$d(x,y) = \|x - y\|$$

is a metric.

$\to$ everything from part A metric spaces still applies.

Recall:

- $x_n \to x \;:\Longleftrightarrow\; \forall \varepsilon > 0 \; \exists N$ s.t. $\forall n \geq N$ $\|x - x_n\| < \varepsilon$.

- (x_n) is Cauchy sequence (C.S.) iff $\forall \varepsilon > 0 \; \exists N$ s.t. $\forall n, m \geq N$
  $\|x_n - x_m\| < \varepsilon$

- $f : (X, \|\_\|_X) \to (Y, \|\_\|_Y)$ is continuous if $\forall x \in X \;\; \forall \varepsilon > 0 \; \exists \delta > 0$ s.t. $\forall \tilde{x} \in X$ with $\|x - \tilde{x}\|_X < \delta$ we have $\|f(x) - f(\tilde{x})\|_Y < \varepsilon$.

Equivalent to:
$\forall (x_n) \subset X$ s.t. $\exists x \in X$ $x_n \to x$ in $(X, \|\_\|_X)$ we have $f(x_n) \to f(x)$ in $(Y, \|\_\|_Y)$.

- $\Omega \subset X$ is open iff $\forall x_0 \in \Omega \;\; \exists \delta > 0$ s.t. $B_\delta(x_0) \subset \Omega$

$$B_\delta(x_0) = \{x \in X : \|x - x_0\| < \delta\}.$$

- F closed iff F^c open
  $\Longleftrightarrow \forall (f_n) \subset F$ s.t. $\exists x \in X$ with $f_n \to x$ we have $x \in F$.

Recall: $\|-\|$, $\|-\|'$ are equivalent

if $\exists C>0$ s.t. $\forall x \in X$

$$C^{-1} \cdot \|x\|' \leq \|x\| \leq C \cdot \|x\|'.$$

⌜equiv. to $\exists C_{1,2} > 0$ s.t. $\forall x \in X$

$$\|x\| \leq C_1 \cdot \|x\|' \quad (1)$$

$$\|x\|' \leq C_2 \cdot \|x\| \quad (2) \quad ⌟$$

Note: If (1) holds then

if (x_n) c.s. $x_n \to x$ in $(X, \|-\|')$

we also have (x_n) is c.s. in $(X, \|-\|)$

$$\|x_n - x\| \leq C_1 \cdot \|x_n - x\|' \longrightarrow 0$$

so also $x_n \to x$ in $(X, \|-\|)$.

So if $\|-\|$, $\|-\|'$ are equiv.

then

$$(X, \|-\|) \text{ Banach}$$

$$\Longleftrightarrow$$

$$(X, \|-\|') \text{ Banach.}$$

⚠ WRONG for Hilbertspaces.

⌜e.g. $\|-\|_2$, innerprod $(x,y) = \sum x_i y_i$

$$\|x\| = |x_1| + \ldots + |x_n|$$ not ind. by any innerprod.

in $x \in \mathbb{R}^n$ ⌟

Def. 2: A normed space $(X, \|-\|)$ is called a Banach space if it is complete i.e. if every C.S. converges.

Def. 3: A innerproduct space $(X, (\cdot,\cdot))$ is called a Hilbert space if it is complete, wrt induced norm
$$\|x\| := \sqrt{(x,x)}.$$

Hilbert space

$\Downarrow$

Banach space

$\Downarrow$

complete metric space

⚠ Our intuition is based on "living" in

- finite dimension
- situation of innerproduct space.

Finite dim:	∞-dim:
• Every linear map is continuous	X
• $T: X \to X$ invertible $\Leftrightarrow$ injective $\Leftrightarrow$ surjective	X
• $F \subset X$ is compact iff closed & bounded	X

Note if $(X, \|\cdot\|_X)$ is n.s.

and $Y \subset X$ any subspace

$\rightsquigarrow$ can turn Y into a n.s.

by just using the norm from X

$$\underset{\substack{\uparrow \\ X}}{\|y\| = \|y\|_X}, \quad (Y, \|\cdot\|) \text{ n.s.}$$

<u>Prop. I.1:</u>

Let $(X, \|\cdot\|_X)$ Banach space,

$Y \subset X$ subspace. Then

$(Y, \|\cdot\|_X)$ is Banach space $\iff$ $Y \subset X$ is closed.

<u>Remark:</u>

The "$\Leftarrow$" direction holds also if $(X, \|\cdot\|_X)$ is not complete, while "$\Rightarrow$" direction is wrong for incomplete spaces, because e.g.

<u>Proof:</u>

"$\Rightarrow$" Let $(y_n) \subset Y$ s.t. $\exists x \in X$ with $y_n \to x$ in $(X, \|\cdot\|_X)$

To show $x \in Y$.

<u>Note:</u> $\|y_n - y_m\|_X \overset{\Delta}{\leq} \|y_n - x\| + \|x - y_m\| \xrightarrow[n,m\to\infty]{} 0$

so (y_n) is C.S. in $(Y, \|\cdot\|_X)$

so, as $(Y, \|\cdot\|_X)$ complete, converges i.e. $\exists y \in Y$ s.t.

$$\|y_n - y\|_X \to 0$$

so $x = y$ by uniq. of limits

(as we have same norm so

$$\|x-y\|_X \leq \|x-y_n\|_X + \|y_n - y\|_X \to 0$$

)

so $x \in Y$ $\boxed{\text{///}}$

"$\Leftarrow$" Let (y_n) be C.S. in $(Y, \|\cdot\|_X)$

so Also (y_n) is C.S. in $(X, \|\cdot\|_X)$

(as same norm and $Y \subset X$)

so by completeness of $(X, \|\cdot\|_X)$

get that $\exists x \in X$ s.t.

$y_n \to x$ in $(X, \|\cdot\|_X)$

As $Y \subset X$ closed, must have

$x \in Y$ so $y_n \to x$ in $(Y, \|\cdot\|_X)$

$\boxed{\text{//}}$