I.2 Examples of normed spaces

$(\mathbb{R}^n, \|\_\|_p)$, $(\ell^p, \|\_\|_p)$, $(L^p, \|\_\|_{L^p})$

Let $1 \leq p \leq \infty$.

- For any $n \in \mathbb{N}$ we can define a norm on $\mathbb{R}^n$ or $\mathbb{C}^n$ by setting

$$\|x\|_p := \left(\sum_{j=1}^{n} |x_j|^p \right)^{1/p} \text{ if } p < \infty$$

resp.

$$\|x\|_\infty := \max_{j \in \{1, \ldots, n\}} |x_j|$$

Note for $0 < p < 1$, above expressions are well defined but violate Δ-ineq.
$\rightarrow$ don't define a norm.

- seq. spaces

$$\ell^p(\mathbb{F}) = \ell^p := \{ (x_n) : x_n \in \mathbb{F} \text{ and } \sum_{j=1}^{\infty} |x_j|^p < \infty \}$$

$p < \infty$, resp

$$\ell^\infty := \{ (x_n) : \text{bounded} \}.$$

equipped with

$$\|(x_n)\|_{\ell^p} := \left(\sum_{j=1}^{\infty} |x_j|^p \right)^{1/p} \quad p < \infty$$

$$\|(x_n)\|_{\ell^\infty} := \sup_{j \in \mathbb{N}} |x_j|$$

- L^p-spaces of functions:

Let $\Omega \subset \mathbb{R}^n$ (measurable)

For $1 \leq p < \infty$

$$\mathcal{L}^p(\Omega) := \{ f : \Omega \to \mathbb{R} \text{ meas. s.t. } \int_\Omega |f|^p dx < \infty \}$$

⌜Remark: If $g \geq 0$, measurable
$\to \int_\Omega g\, dx$ well def. but might be $+\infty$ ⌟

$$\mathcal{L}^\infty(\Omega) := \{ f : \Omega \to \mathbb{R} \text{ meas. "essentially bounded" i.e. } \exists M \text{ s.t. } |f| \leq M \text{ a.e.} \}.$$

Define

$$\|f\|_{\mathcal{L}^p} := \left(\int_\Omega |f|^p dx \right)^{1/p}, \quad f \in \mathcal{L}^p$$

$$\|f\|_{\mathcal{L}^\infty} := \operatorname{ess\,sup}_\Omega |f| = \inf\{ M : |f| \leq M \text{ a.e.} \}.$$

<u>Note:</u> (N2) & (Δ) & $\|f\| \geq 0$ ok
but $\|f\|_{\mathcal{L}^p} = 0$ only implies that $f = 0$ a.e.

Set $L^p(\Omega) := \mathcal{L}^p(\Omega) / \sim$
equipped with $\|\text{-}\|_{\mathcal{L}^p}$
where we identify $f \sim g$ if $f = g$ a.e. and get that
$(L^p(\Omega), \|\text{-}\|_{L^p})$ is normed space.

Remark:

- All norms $\|\cdot\|_p$ (indeed all norms) on $\mathbb{F}^n$ are equivalent.
  ($\mapsto$ ch. 3 Finitedim spaces)

  and $(\mathbb{F}^n, \|\cdot\|_p)$ are Banach.

⚠ WRONG for ∞ dim "version" < seq. space / function space

- For seq. spaces we have

$$\ell^p \subsetneqq \ell^q \text{ if } p < q$$

- For $\Omega \subset \mathbb{R}^n$ which $\mathcal{L}^n(\Omega) < \infty$ (e.g. Ω bounded) we have

$$L^p(\Omega) \subsetneqq L^q(\Omega) \text{ if } p > q$$

⚠ Reverse inequalities needed for ℓ^p, L^p

Trick to remember this:

- $(x_n) \in \ell^1$, i.e. $\sum |x_n| < \infty \Rightarrow x_n \to 0$
  $\Rightarrow |x_n|$ bounded

  so $\ell^1 \subset \ell^\infty$

- $\mathcal{L}^n(\Omega) < \infty$, f bounded $\Rightarrow$ finite integr.

  so $L^\infty(\Omega) \subset L^1(\Omega)$.

Example of non-complete normed space:

$(L^\infty([0,1]), \|-\|_{L^1})$

is normed space, we can see

as a subspace of $(L^1([0,1]), \|-\|_{L^1})$

with induced norm.

By Prop. I.1 (& following remark)

if $(L^\infty(\overset{I}{\overset{\shortparallel}{[0,1]}}), \|-\|_{L^1})$ was complete

then $L^\infty(I) \subset L^1(I)$ closed.

However take e.g. $f(x) = \frac{1}{\sqrt{x}} \in L^1(I) \setminus L^\infty(I)$

we have that

$$f_n(x) = \frac{1}{\sqrt{x}} \mathbb{1}_{[\frac{1}{n},1]} \in L^\infty$$

and

$$\|f - f_n\|_{L^1} = \int_0^{1/n} \frac{1}{\sqrt{x}}\,dx = 2 \cdot \sqrt{\tfrac{1}{n}} \xrightarrow[n\to\infty]{} 0$$

so $f_n \to f$ in L^1, so $L^\infty \subset L^1$

not closed, so $(L^\infty, \|-\|_{L^1})$

not a Banach space.

Very useful result for dealing with ℓ^p, L^p:

Lemma I.2 (Hölder's inequality)

Let $1 \leq p, q \leq \infty$ be conjugate exponents, i.e. $\frac{1}{p} + \frac{1}{q} = 1$ $\left(\frac{1}{\infty} = 0\right)$

Then:

- $\forall x, y \in \mathbb{R}^n$ we have
$$\left|\sum_{i=1}^{n} x_i y_i\right| \leq \|x\|_p \cdot \|y\|_q$$
- $\forall (x_n) \in \ell^p$, $\forall (y_n) \in \ell^q$ the series $\sum_{j=1}^{\infty} x_j y_j$ converges absolutely and
$$\left|\sum_{j=1}^{\infty} x_j y_j\right| \leq \|(x_n)\|_{\ell^p} \|(y_n)\|_{\ell^q}$$
- If $f \in L^p(\Omega)$, $g \in L^q(\Omega)$ then $f \cdot g$ is integrable and
$$\left|\int_\Omega f \cdot g \, dx\right| \leq \|f\|_{L^p} \cdot \|g\|_{L^q}$$

⌜Also true that
$$\sum |x_i y_i| \leq \dots$$
$$\int |f \cdot g| \leq \dots$$
as we can replace (x_n) by $(|x_n|)$, f by $|f|$ without changing $\|-\|_{\ell^p}$ resp. $\|-\|_{L^p}$. ⌟

Remark:

For $p=2$ have that norms are induced by following inner products:

- $\mathbb{F}^n$ $\quad (x,y) = \sum_{j=1}^{n} x_j \overline{y_j}$
- $(\ell^2, \|-\|_2)$, $\quad ((x_n),(y_n)) = \sum_{j=1}^{\infty} x_j \overline{y_j}$

  where Hölder ($p=q=2$) guarantees that series converges
- $(L^2, \|-\|_{L^2})$,

  $$(f,g)_{L^2} := \int_\Omega f \cdot \overline{g}\, dx$$

  where $f \cdot \overline{g}$ is integrable by Hölder.

A small applic. of Hölder ($\rightarrow$ Q2 on sheet 1)

Claim: $L^4([0,2]) \subset L^2([0,2])$

and $\|f\|_{L^2} \leq \sqrt[4]{2} \cdot \|f\|_{L^4}$

Proof: Let $f \in L^4([0,2])$

so $\int (|f|^2)^2 < \infty$ ie. $|f|^2 \in L^2$

So by Hölder we get

$$\|f\|_{L^2}^2 = \int_I |f|^2 \cdot 1 \leq \||f|^2\|_{L^2} \cdot \underbrace{\|1\|_{L^2}}_{= \left(\int_0^2 1 dx\right)^{\frac{1}{2}} = \sqrt{2}}$$

$$= \left(\int |f|^4 dx\right)^{1/4 \cdot 2} \cdot \sqrt{2}$$

$$\|f\|_{L^2} \leq \sqrt[4]{2}\, \|f\|_{L^4} \qquad \blacksquare$$

Functionspaces with supremum norm:

Given a space of functions which are bounded, such as

$$\mathcal{F}^b(\Omega) := \{f : \Omega \to \mathbb{R}\ (\mathbb{C}),\ \text{bounded}\}$$

or

$$C_b(\Omega) := \{f : \Omega \to \mathbb{R}\ (\mathbb{C}) \text{ continuous and bounded}\}$$

we can equip these spaces with

$$\|f\|_{sup} := \sup_{x \in \Omega} |f(x)|$$

<u>Note</u>: If $\Omega \subset \mathbb{R}^n$ compact then $C(\Omega) = C_b(\Omega)$.

More generally e.g. for Ω comact can equip

$$C^1(\Omega) = \{f : \Omega \to \mathbb{R} \text{ continuously differentiable}\}$$

with

$$\|f\|_{C^1} = \|f\|_{sup} + \|f'\|_{sup} \quad \text{if } \Omega \subset \mathbb{R}$$

resp

$$\|f\|_{C^1} = \|f\|_{sup} + \sum_{j=1}^{n} \|\partial_{x_j} f\|_{sup} \quad \text{if } \Omega \subset \mathbb{R}^n$$

Turns out that this is "right" choice as $(C^1(\Omega), \|\_\|_{C^1})$ is complete.

On the other hand $(C^1(\Omega), \|\_\|_{sup})$ incomplete as $C^1(\Omega) \subset C^0(\Omega)$ not closed, e.g. $\underset{\in C^0 \setminus C^1}{f(x) = |x|}$, $\underset{\in C^1}{f_n(x) = \left(x^2 + \frac{1}{n}\right)^{1/2}}$

A few more examples/constructions:

- Quotions of (vector space, seminorm)

Let X vector space

$\|-\|_0$ seminorm, ≥ 0 ✓

(N2) ✓

Δ ✓

then $(X/X_0, \|-\|)$ is a normed space where $X_0 = \{x : \|x\|_0 = 0\}$

and $\|[x]\| = \|x\|_0$

$\to$ problem sheet.

- $(X, \|-\|_X)$, $(Y, \|-\|_Y)$ normed spaces

then can define norm on $X \times Y$ by

$$\|(x,y)\| = \left(\|x\|_X^p + \|y\|_Y^p\right)^{1/p} \quad 1 \leq p < \infty$$

or $\|(x,y)\| = \max(\|x\|_X, \|y\|_Y)$

<u>Note:</u> $(X \times Y, \|-\|)$ "inherits" nice properties of $(X, \|-\|_X)$, $(Y, \|-\|_Y)$ e.g.

if ---, --- are Banach

also $(X \times Y, \|-\|)$ is Banach.

<u>Also:</u> If $(X, \|-\|_X)$, $(Y, \|-\|_Y)$ are Hilbert spaces

then $p=2$ gives $(X \times Y, \|-\|)$ is Hilbert space.

- If $X_{1,2}$ subspaces of $(X, \|-\|_X)$

then $X_1 + X_2$ is subspace so $(X_1 + X_2, \|-\|_X)$ normed space, but

⚠ $(X_1, \|\cdot\|_X), (X_2, \|\cdot\|_X)$

Banach

$\not\Rightarrow$ $(X_1 + X_2, \|\cdot\|_X)$ Banach.

→ p.s.: Examples where $X_{1,2} \subset X$

are closed

but $X_1 + X_2 \subset X$

NOT closed.