Last videos:

- Def: Banach space = complete normed space
- seen various examples of normed spaces
  - $(\mathbb{F}^n, \|-\|_p)$, $(\ell^p, \|-\|_p)$, $(L^p, \|-\|_{L^p})$
  - $(C_b, \|-\|_{sup})$, ...

This video:

How to prove completeness?

Rough heuristics:

Given a space X

- If we define a norm which is not "strong enough" then we get
  - well def. norm
  - but $(X, \|-\|)$ not complete

  (e.g. $(L^\infty, \|-\|_{L^1})$)

- If we try to define a "too strong object" as norm we might end up considering a quantity that is NOT WELL DEFINED on X,

  e.g. $(L^1, \|-\|_{L^\infty})$ $\times$

  will be ∞ for some elements of L^1

Ex. 1: $(C_b(\Omega), \|-\|_{sup})$ is Banach space, $\Omega \subset \mathbb{R}^n$.

Easy to see: $\|-\|_{sup}$ is norm on $C_b(\Omega)$

so all ~~we~~ we need to show is completeness.

Let (f_n) C.S. in $(C_b(\Omega), \|-\|_{sup})$

1. Given any $x \in \Omega$ we have that $(f_n(x))$ is C.S. in $\mathbb{F}$ ($= \mathbb{R}$ or $\mathbb{C}$) since

$$|f_n(x) - f_m(x)| \leq \|f_n - f_m\| \xrightarrow[n,m\to\infty]{} 0.$$

So by completeness of $\mathbb{F}$ we get that $\exists f(x) \in \mathbb{F}$ s.t.

$$f_n(x) \xrightarrow[n\to\infty]{} f(x).$$

2. Claim: $f \in C_b(\Omega)$, $\|f_n - f\|_{sup} \to 0$.

Proof: Let $\varepsilon > 0$.

As (f_n) C.S. in $C_b(\Omega)$ so $\exists N \in \mathbb{N}$ s.t. $\forall n, m \geq N$ have

$$\sup_{x\in\Omega} |f_n(x) - f_m(x)| < \varepsilon.$$

Given any $x \in \Omega$ have $\forall n \geq N$

$$|f_n(x) - f(x)| = |f_n(x) - \lim_{m\to\infty} f_m(x)|$$
$$= \lim_{m\to\infty} |f_n(x) - f_m(x)|$$
$$\leq \varepsilon$$

so get $\forall x \in \Omega$

- $|f(x)| \leq \|f_N\|_{sup} + \varepsilon$

  so f is bounded

- $\|f_n - f\|_{sup} \leq \varepsilon$

  and $\varepsilon > 0$ was arbitrary

  get $\|f_n - f\|_{sup} \to 0$

  so $f_n \to f$ uniformly

  and thus f is continuous

  as uniform limit of continuous

  functions ($\to$ Analysis II)

$\boxed{///}$

<u>Ex. 2</u>: $(\ell^2, \|\cdot\|_{\ell^2})$ complete

<u>Proof</u>: Let $(x^{(m)}) \subset \ell^2$ C.S.

$x^{(m)} = (x_j^{(m)})_{j \in \mathbb{N}}$

1

Let $j \in \mathbb{N}$.

Then $(x_j^{(m)})_{m \in \mathbb{N}}$ is C.S. in $\mathbb{F}$

as
$$|x_j^{(m)} - x_j^{(n)}|^2 \leq \sum_{k=1}^{\infty} |x_k^{(m)} - x_k^{(n)}|^2$$
$$= \|x^{(m)} - x^{(n)}\|_{\ell^2}^2$$
$$\xrightarrow[m,n \to \infty]{} 0.$$

So by completeness of $\mathbb{F}$ we have

$\exists\, x_j \in \mathbb{F}$ s.t. $x_j^{(m)} \xrightarrow[m \to \infty]{} x_j$.

(2) Claim: $x = (x_j) \in \ell^2$

and $\|x - x^{(n)}\|_{\ell^2} \to 0$

Use "standard abuse of notation"

$$\|(y_j)\|_{\ell^2} = \begin{cases} \left(\sum_{j=1}^{\infty} |y_j|^2\right)^{1/2} & (y_j) \in \ell^2 \\ \infty & \text{else}. \end{cases}$$

Note: Still have that $\forall$ seq.

$(x_j), (y_j)$

$\|(x_j) + (y_j)\|_{\ell^2} \le \|(x_j)\|_{\ell^2} + \|(y_j)\|_{\ell^2}$

because either r.h.s $= +\infty$ so ✓

or both seq. $\in \ell^2 \to$ standard

Δ-ineq. for space ℓ^2.

Proof of claim:

Let $\varepsilon > 0$, let N s.t. $\forall n,m \geq N$

$$\sum_{j=1}^{\infty} |x_j^{(n)} - x_j^{(m)}|^2 = \|x^{(n)} - x^{(m)}\|_{\ell^2}^2 < \varepsilon^2.$$

⚠ AOL is WRONG for ∞ sums!

Let $K \in \mathbb{N}$ any number and note

$$\sum_{j=1}^{K} |x_j^{(n)} - x_j^{(m)}|^2 < \varepsilon^2$$

for $n,m \geq N$

so by AOL

$$\sum_{j=1}^{K} |x_j^{(n)} - x_j|^2 = \lim_{m\to\infty} \sum_{j=1}^{K} |x_j^{(n)} - x_j^{(m)}|^2 \leq \varepsilon^2$$

$n \geq N$

so sending $K \to \infty$ we get

$$\|x^{(n)} - x\|_{\ell^2}^2 \leq \varepsilon^2$$

Note in particular $x \in \ell^2$ since

$$\|x\|_{\ell^2} \overset{\triangle}{\leq} \|x^{(N)} - x\|_{\ell^2} + \|x^{(N)}\|_{\ell^2} \leq \varepsilon + \|x^{(N)}\|_{\ell^2} < \infty.$$

and as $\varepsilon > 0$ was arbitrary get $\|x^{(n)} - x\|_{\ell^2} \to 0$

▥ claim

▥ completeness of $(\ell^2, \|\cdot\|_{\ell^2})$.

Remark:

$(L^p(\Omega), \|-\|_{L^p})$, $\Omega \subset \mathbb{R}^n$

$1 \leq p \leq \infty$

is Banach space.

Useful fact & warning:

Let $(f_n) \subset L^p(\Omega)$

Then:

If $f_n \to f$ a.e. $\not\Rightarrow$ $f_n \to f$ in L^p

but

$f_n \to f$ in L^p $\Rightarrow$ $\exists$ subseq. s.t. $f_{n_j} \to f$ a.e.

Useful results to prove completeness:

Lemma I.3:

Let (x_n) C.S. in $(X, \|\cdot\|)$.

Then:

x_n converges $\iff$ $\exists$ subsequence x_{n_j} which converges.

Proof: "$\Rightarrow$" trivial

"$\Leftarrow$" Let (x_n) C.S., suppose $x_{n_j} \to x$.

Let $\varepsilon > 0$. Then

- $\exists J \in \mathbb{N}$ s.t. $\forall j \geq J$ we have $\|x_{n_j} - x\| < \varepsilon/2$
- $\exists N \in \mathbb{N}$ s.t. $\forall n, m \geq N$ $\|x_n - x_m\| < \varepsilon/2$.

Given any $n \geq N$ take some j s.t. $j \geq J$ and $n_j \geq N$ get

$$\|x - x_n\| \overset{\triangle}{\leq} \underbrace{\|x - x_{n_j}\|}_{< \varepsilon/2} + \underbrace{\|x_{n_j} - x_n\|}_{< \varepsilon/2} < \varepsilon$$

$\blacksquare$

Lemma I.4:

Let $(X, \|\cdot\|)$ n.s. Then:

$(X, \|\cdot\|)$ is Banach $\Longleftrightarrow$ Absolute converg. implies converg.,

i.e. $\forall (x_n) \subset X$ with $\sum_{j=1}^{\infty} \|x_j\| < \infty$ we have $S_n := \sum_{j=1}^{n} x_j$ converges in $(X, \|\cdot\|)$.

Proof:

"$\Rightarrow$" Let (x_n) s.t. $\sum_{j=1}^{\infty} \|x_j\| < \infty$

$$S_n = \sum_{j=1}^{n} x_j.$$

Then: $m \geq n$

$$\|S_n - S_m\| = \left\| \sum_{j=n+1}^{m} x_j \right\| \overset{\triangle}{\leq} \sum_{j=n+1}^{\infty} \|x_j\| \xrightarrow[n\to\infty]{} 0$$

so (S_n) is C.S. in $(X, \|\cdot\|)$

so (S_n) converges as $(X, \|\cdot\|)$ is complete $\boxed{}$

"$\Longleftarrow$" Let (x_n) C.S.

Let (x_{n_j}) be a subsequence s.t.

$$\|x_{n_j} - x_{n_{j+1}}\| \leq 2^{-j}$$

Then $\sum \|x_{n_j} - x_{n_{j+1}}\| < \infty$

so by assumption

$$S_m := \sum_{j=1}^{m} (x_{n_{j+1}} - x_{n_j}) = x_{n_{m+1}} - x_{n_1}$$

converges.

So

$$x_{n_{m+1}} = S_m + x_{n_1}$$

converges and we have found a conv. subsequence (x_{n_j})

By Lemma I.3 thus (x_n) converges. $\boxed{\text{///}}$

Example of non-complete space:

$(C^0(I), \|-\|_{L^1})$ not complete.

$I = [0,1]$

Possible ways of proving this:

① Use Lemma I.4 and construct a sequence $(f_n) \subset C^0(I)$

s.t. $\sum_{n=1}^{\infty} \|f_n\|_{L^1} < \infty$

but $\sum_{j=1}^{n} f_j$ can't converge

to a $f \in C^0(I)$. → lecture notes

② Using that $C^0(I) \subset L^1(I)$ not closed

proving

E.g. construct $(f_n) \subset C^0(I)$ s.t.

$f_n \to f$ in $(L^1, \|\cdot\|_{L^1})$

for some $f \in L^1 \setminus C^0$

(i.e. element of L^1 s.t. $\nexists$ representative which is continuous)

or later in course we will see that $C^0(I)$ is <u>dense</u> in $L^1(I)$

i.e. $\overline{C^0(I)}^{L^1} = L^1(I)$.

So if $C^0(I)$ was closed then

$$C^0(I) = \overline{C^0(I)}^{L^1} = L^1(I)$$

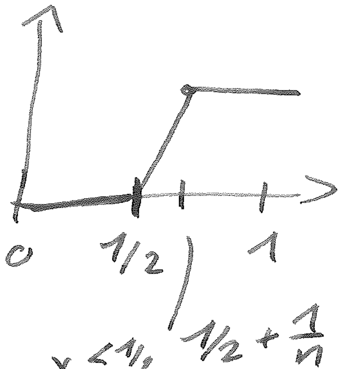
so show $L^1(I) \setminus C^0(I) \neq \emptyset$

e.g. $I = [0,1]$, $f = \mathbb{1}_{[1/2, 1]}$ $\in L^1 \setminus C^0$.

③ Direct argument.

Construct a seq. (f_n) which is c.s. but can't converge.

Let f_n



$$f_n(x) = \begin{cases} 0 & x \leq 1/2 \\ \text{linear} & \text{- - - -} \\ 1 & x \geq \frac{1}{2} + \frac{1}{n} \end{cases}$$

<u>Know</u>: $f_n \to f = \begin{cases} 0 & [0, 1/2] \\ 1 & (1/2, 1] \end{cases}$

a.e.

so every subseq. $f_{n_j} \to f$ a.e.

If (f_n) was convergent in $(C^0, \|\cdot\|_{L^1})$

say $\exists \tilde{f} \in C^0(I)$ s.t.

$$\|\tilde{f} - f_n\|_{L^1} \to 0$$

then $\exists$ subsequence $f_{n_j} \to \tilde{f}$ a.e.

so would get $f = \tilde{f}$ a.e.

Impossible as $\tilde{f}$ continuous.

$\to$ (f_n) can't converge in $(C^0, \|\cdot\|_{L^1})$

Claim: (f_n) is C. S. in - - -

Proof: • $f_n \in C^0(I)$.

• $n, m \geq N$, $f_n = f_m$ outside $[1/2, 1/2 + \frac{1}{N}]$

so as $0 \leq f_{n,m} \leq 1$ we get

$$\|f_n - f_m\|_{L^1} \leq \int_{1/2}^{1/2+\frac{1}{N}} 1 \, dx \leq \frac{1}{N} \xrightarrow[N \to \infty]{} 0$$

$\boxed{\square}$