Chapter II

Bounded linear operators

<u>Def. 4:</u> Let $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$ be any n.s.

$$T : (X, \|\cdot\|_X) \to (Y, \|\cdot\|_Y)$$

is a <u>bounded linear operator</u>

if
- $T: X \to Y$ is linear, i.e.

  $\forall x, \tilde{x} \in X \ \forall \lambda \in \mathbb{F}$

  $$T(x + \lambda \tilde{x}) = Tx + \lambda T\tilde{x}$$

and
- $\exists M \in \mathbb{R}$ s.t.

  $$\|Tx\|_Y \leq M \cdot \|x\|_X . \quad (*)$$

Define

$$\underset{\underset{\mathcal{B}(X,Y)}{\|}}{\overset{\overset{L((X,\|\cdot\|_X),(Y,\|\cdot\|_Y))}{\|}}{L(X,Y)}} := \{ T \text{ bounded linear operator from } (X, \|\cdot\|_X) \text{ to } (Y, \|\cdot\|_Y) \}.$$

which we will always equip with the "operator norm"

$$\|T\|_{L(X,Y)} := \inf\{ M \geq 0 \text{ satisfying } (*) \}.$$

Special case:
- If $(Y, \|\cdot\|_Y) = (X, \|\cdot\|_X)$ write $L(X) = L(X,X)$
- If $(Y, \|\cdot\|_Y) = (\mathbb{F}, |\cdot|)$ write $X^* = L(X, \mathbb{F})$

Important remark:

- $\|\cdot\|_{L(X,Y)}$ is a norm on $L(X,Y)$ and it's the only norm on $L(X,Y)$ we'll use in this course.

- For $X \neq \{0\}$ we have that for $T \in L(X,Y)$

$$\|T\| = \sup_{\substack{x \in X \\ x \neq 0}} \frac{\|Tx\|_Y}{\|x\|_X} = \sup_{\substack{\|x\| \leq 1 \\ x \in X}} \|Tx\|_Y .$$

Proof of $\|T\| = \sup_{x \in X \setminus \{0\}} \frac{\|Tx\|_Y}{\|x\|_X}$.

Let M be any number $M \geq 0$ s.t.

$$(*) \quad \|Tx\|_Y \leq M \cdot \|x\|_X$$

Get $\frac{\|Tx\|_Y}{\|x\|_X} \leq M \quad \forall x \in X \setminus \{0\}$

so $K := \sup_{x \in X \setminus \{0\}} \frac{\|Tx\|_Y}{\|x\|_X} \leq M$

And thus

$$K \leq \inf\{M : (*) \text{ holds}\} = \|T\|$$

Trivially we have $\forall x \in X$

$$\|Tx\|_Y \leq K \cdot \|x\|_X$$

so K sat. $(*)$ so $\|T\| \leq K$ $\square$

Remark:

We have $\|Tx\|_Y \leq \|T\|_{L(X,Y)} \|x\|_X$

$$\forall x \in X$$

i.e. infimum in the definition of the operator norm is actually achieved.

⚠ In general there won't be any $x \in X$ s.t.

$$\frac{\|Tx\|_Y}{\|x\|_X} = \|T\|$$

i.e. $\sup \frac{\|Tx\|_Y}{\|x\|_X} = \|T\|$

is in general not achieved.

⚠ $T(X)$ is not a bounded set, unless $T \equiv 0$, however

if T bounded linear op. then Images of bounded subsets $\Omega \subset X$ will be bounded subsets of Y

$\ulcorner$ $\Omega \subset B_R(0) \subset X$

$$\Rightarrow T(\Omega) \subset B_{\|T\|R}(0) \subset Y. \lrcorner$$

Prop. II. 1:

Let $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$ be n.s., let $T: X \to Y$ linear map.

Then the following are equivalent:

(i) T Lipschitz continuous on all of X

(ii) T is continuous on X

(iii) T is continuous at 0

(iv) T is a bounded linear operator.

Proof: (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii)

(iii) $\Rightarrow$ (iv) By contradiction:

Let $T: X \to Y$

assume
- T continuous at 0
- T NOT a bounded lin. op.

So $\forall n \in \mathbb{N}$ $\exists x_n$ s.t.

$$\|Tx_n\|_Y \cancel{\leq} n \cdot \|x_n\|_X$$

i.e. s.t. $\|Tx_n\|_Y > n \cdot \|x_n\|_X \geq 0$

So $\tilde{x}_n := \frac{x_n}{\|Tx_n\|_Y}$ are s.t.

$$\|\tilde{x}_n\| = \frac{\|x_n\|}{\|Tx_n\|} < \frac{1}{n} \to 0$$

So $\tilde{x}_n \to 0$ and thus as T continuous at 0 we get $T\tilde{x}_n \to T0 \overset{\text{lin}}{=} 0$, i.e.

$$\|T\tilde{x}_n\| \to 0$$

$$\|T\tilde{x}_n\| = \left\| \frac{Tx_n}{\|Tx_n\|} \right\| = 1 \quad \lightning$$

(iv) $\Rightarrow$ (i)

Let $x, \tilde{x} \in X$

$$\|Tx - T\tilde{x}\| \underset{\substack{\uparrow \\ \text{linearity}}}{=} \|T(x-\tilde{x})\|$$

$$\leq \|T\| \cdot \|x-\tilde{x}\|$$

so T Lip. cont. with Lip const. $\|T\|$.

$\blacksquare$