II. Bounded linear operators

Last time

- $T \in L(X,Y)$ iff linear and $\exists M$ s.t. $\|Tx\|_Y \leq M \cdot \|x\|_X \quad \forall x \in X$
- $\|T\| := \inf\{\text{such } M\text{'s}\} = \sup \dfrac{\|Tx\|_Y}{\|x\|_X}$

Have seen: $T \in L(X,Y)$

$\iff$

T linear & continuous

Today: • How to determine
- if $T \in L(X,Y)$
- $\|T\|$

To prove that a given T is an element of $L(X,Y)$:

- T is well defined as $T: X \to Y$, i.e. $T(x) \in Y \quad \forall x \in X$
- T is linear ("obviously", or by "linearity of $\int \dots$")
- Find some M (does not need to be optimal) s.t.

$$\|Tx\|_Y \leq M \cdot \|x\|_X \quad \forall x \in X \quad (*)$$

To also determine $\|T\|$:

- Find some candidate M for $\|T\|$ and check $(*)$ holds

$\rightarrow \quad \|T\| \leq M.$

Show that

$$\|T\| \geq M$$

by either finding $x \in X$ s.t.

$$\frac{\|Tx\|}{\|x\|} = M$$

(won't work in general)

or find seq. $(x_n) \subset X$ s.t.

$$\frac{\|Tx_n\|}{\|x_n\|} \to M$$

resp. show that $\forall \varepsilon > 0 \ \exists x$

s.t. $\dfrac{\|Tx\|}{\|x\|} \geq M - \varepsilon$.

Ex. 1 : Shifts & projections on ℓ^p.

Given $1 \le p \le \infty$.

Can consider

$$L, R : \ell^p \to \ell^p$$

$$\pi_k : \ell^p \to \mathbb{F}$$

defined by

$$L((x_1, x_2, \dots)) = (x_2, x_3, \dots)$$

$$R((x_1, x_2, \dots)) = (0, x_1, x_2, \dots)$$

$$\pi_k((x_j)) = x_k$$

<u>Claim:</u> $L, R \in L(\ell^p, \ell^p)$

$\pi_k \in (\ell^p)^*$

with $\|L\| = \|R\| = \|\pi_k\| = 1$.

<u>Proof:</u> Obviously L, R, π_k are linear

Also $\forall x \in \ell^p$:

$\|L(x)\|_{\ell^p} \le \|x\|_{\ell^p} \stackrel{<\infty}{}$ so $L \in L(\ell^p)$

with $\|L\| \le 1$

$\|R(x)\|_{\ell^p} = \|x\|_{\ell^p}$ so $R \in L(\ell^p)$

with $\|R\| = 1$

and indeed R is isometric i.e. "preserves norm" for all elements of X.

$|\pi_k(x)| \le \|x\|_{\ell^p}$, $\pi_k \in (\ell^p)^*$

with $\|\pi_k\| \le 1$.

Let $e^{(k)} = (0, 0, \dots, 0, \underset{\substack{\uparrow \\ k\text{th position}}}{1}, 0, \dots)$

Then $\dfrac{|\pi_k(e^{(k)})|}{\|e^{(k)}\|_{\ell^p}} = \dfrac{1}{1} = 1$ so $\|\pi_k\| \ge 1$.

and for $k \ge 1$ $L(e^{(k)}) = e^{(k-1)}$

so $\dfrac{\|L(e^{(k)})\|}{\|e^{(k)}\|} = \dfrac{1}{1} \to \|L\| \ge 1$ ☒

Ex 2: Multiplication by a function

on $X = C(I)$, say $I = [0,1]$

with sup-norm.

Let $g \in C(I)$ consider

$T : f \mapsto Tf$, $(Tf)(x) = f(x)g(x)$

Claim: $T \in L(X, X) = L(X)$

and $\|T\| = \|g\|_{sup}$

Proof: As product of continuous functions is continuous

have $T : X \to X$ welldef.

Also, clearly T is linear and

for any $f \in X$ can bound

$$\|Tf\|_{sup} = \sup_{x \in I} |f(x)g(x)|$$

$$\leq \sup_{x \in I} |f(x)| \cdot \sup_{x \in I} |g(x)|$$

$$= \|f\|_{sup} \cdot \|g\|_{sup}$$

so $T \in L(X)$ with $\|T\| \leq \|g\|_{sup}$.

Also for $f \equiv 1$ (const. function), $\in X$,

have $\|f\|_{sup} = 1$ and $Tf = g$

so $\dfrac{\|Tf\|_{sup}}{\|f\|_{sup}} = \|g\|_{sup}$

so $\|T\| \geq \|g\|_{sup}$ so $\|T\| = \|g\|_{sup}$.

$\boxtimes$

Ex. 3 Multiplication by a function on L^p-spaces:

Let $X = L^1(I)$, $I \subset \mathbb{R}$

Let $g \in L^\infty(I)$.

Then $T: f \mapsto Tf$

$$(Tf)(x) = f(x)g(x)$$

is $\in L(L^1)$ and

$$\|T\| = \|g\|_{L^\infty}.$$

<u>Proof</u>: T is welldef. since

- changing f on a nullset only changes $f\cdot g$ on - - - - so get same element of L^1.

- $$\int |Tf(x)|dx = \int |f(x)\cdot g(x)|dx \underset{\text{Hölder}}{\le} \|g\|_{L^\infty}\cdot \|f\|_{L^1}$$

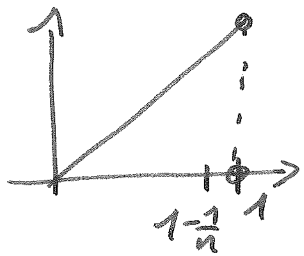
so $Tf \in L^1$ and $\dfrac{\|Tf\|_{L^1}}{\|f\|_{L^1}} \le \|g\|_{L^\infty}$

So $T \in L(L^1)$ and $\|T\| \le \|g\|_{L^\infty}$.

<u>Claim</u>: $\|T\| \ge \|g\|_{L^\infty}$

<u>Proof</u>: <u>Ex</u>: $g(x) = x$

Let $f_n = \mathbb{1}_{[1-\frac{1}{n},1]}$



Note $g \ge 1-\frac{1}{n}$ on set where $f_n \ne 0$

so $1 \ge \dfrac{\|f_n \cdot g\|_{L^1}}{\|f_n\|_{L^1}} \ge \dfrac{\cancel{\frac{1}{n}}(1-\frac{1}{n})}{\cancel{\frac{1}{n}}} = 1-\frac{1}{n}$ so $\|T\| \ge 1 = \|g\|_{L^\infty}$.

For general $g \in L^\infty(I)$

Given any $\varepsilon > 0$ we have that

$$\|g\|_{L^\infty} - \varepsilon < \|g\|_{L^\infty} = \inf\{M : |g| \le M \text{ a.e.}\}$$

So can't have

$$|g| \le \|g\|_{L^\infty} - \varepsilon \text{ a.e.}$$

So $A_\varepsilon := \{x \in I : |g(x)| \geq \|g\|_{L^\infty} - \varepsilon\}$

has $\mathcal{L}(A_\varepsilon) > 0$.

Consider $f = \mathbb{1}_{A_\varepsilon}$, then

$$|gf| \geq (\|g\|_{L^\infty} - \varepsilon) \cdot \mathbb{1}_{A_\varepsilon}$$

so

$$\frac{\|Tf\|_{L^1}}{\|f\|_{L^1}} \geq \frac{(\|g\|_{L^\infty} - \varepsilon) \cdot \cancel{\mathcal{L}(A_\varepsilon)}}{\cancel{\mathcal{L}(A_\varepsilon)}} = \|g\|_{L^\infty} - \varepsilon$$

Holds $\forall \varepsilon > 0$

so $\|T\| \geq \|g\|_{L^\infty}$ $\boxtimes$

Exercise:

Think about what "right" constraint on g is if we want

$$T : L^p \to L^q.$$

(restriction on p, q)