II Bounded linear operators (video 3)

<u>Last time:</u>

examples of bounded linear operators

- shifts / projections on sequence spaces
- multiplication by functions on
  - C^0
  - L^1

Today:
- More examples
- Crucial property of $L(X,Y)$

We'll see: $L(X,Y)$ inherits completeness of Y

Ex 1 of an unbounded operator:

$$L : (\ell^1, \|\cdot\|_{\ell^\infty}) \to (\ell^1, \|\cdot\|_{\ell^1})$$

Then $L : \ell^1 \to \ell^1$ linear

but considering

$$x^{(n)} := (1, \dots, \underset{\substack{\uparrow \\ n}}{1}, 0, 0, \dots)$$

we have $\|x^{(n)}\|_{\ell^\infty} = 1$

$$\|Lx^{(n)}\|_{\ell^1} = \|x^{(n-1)}\|_{\ell^1} = n-1$$

so $$\frac{\|Lx^{(n)}\|_{\ell^1}}{\|x^{(n)}\|_{\ell^\infty}} \xrightarrow[n\to\infty]{} \infty$$

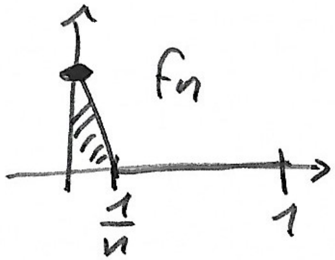
so L is unbounded as operator from $(\ell^1, \|\cdot\|_{\ell^\infty}) \to (\ell^1, \|\cdot\|_{\ell^1})$.

Ex 2 of unbounded operator:

$$T : (C^0([0,1]), \|\cdot\|_{L^1}) \to \mathbb{R}$$

$$Tf := f(0).$$

Note: For



we have

$$\frac{|Tf_n|}{\|f_n\|_{L^1}} = \frac{1}{\frac{1}{2}\cdot\frac{1}{n}} \xrightarrow[n\to\infty]{} \infty$$

so T is unbounded!

Q: What do I ~~m~~ need to ask of a function g s.t.

$$T : L^p(I) \to L^q(I)$$

$$f \mapsto f \cdot g$$

is a bounded linear operator.

A: We want $p > q$

and $g \in L^r(I)$ for

$$\frac{1}{p} + \frac{1}{r} = \frac{1}{q}. \iff \frac{1}{p/q} + \frac{1}{r/q} = 1$$

Indeed need

$$\int |f \cdot g|^q = \int |f|^q \cdot |g|^q$$

$$\overset{\text{Hölder}}{\leq} \| |f|^q \|_{L^{p/q}} \cdot \| |g|^q \|_{L^{r/q}}$$

$$\rightarrow \| f \cdot g \|_{L^q} \leq \|f\|_{L^p} \cdot \|g\|_{L^r}.$$

Ex. 4: Linear maps $T: \mathbb{R}^n \to \mathbb{R}^m$

Note T can be written as

$$(*) \quad Tx = A \cdot x, \quad A \in M_{m \times n}(\mathbb{R})$$

$\uparrow$ vector space.

On $M_{m\times n}(\mathbb{R})$ can consider the Hilbert-Schmidt-norm

$$\|A\|_{HS} := \Big(\sum_{i,j} |a_{ij}|^2\Big)^{1/2}$$

Lemma II. 2:

Any linear map $T: (\mathbb{R}^n, \|\cdot\|_2) \to (\mathbb{R}^m, \|\cdot\|_2)$ is a bounded linear operator and

$$\underset{\uparrow \text{ operator norm}}{\|T\|} \leq \underset{\uparrow \text{ s.t. } (*)}{\|A\|_{HS}}$$

Proof:

Let $x \in \mathbb{R}^n$

$$\|Tx\|_2^2 = \sum_{j=1}^m \big(\underbrace{(Tx)_j}_{=(Ax)_j}\big)^2$$

$$= \sum_{j=1}^m \Big(\sum_{i=1}^n a_{ji} x_i\Big)^2$$

$$\underset{\text{Cauchy-Schwarz}}{\leq} \underbrace{\Big[\sum_{j=1}^m \Big(\sum_i a_{ji}^2\Big)\Big]}_{=\|A\|_{HS}^2} \cdot \underbrace{\Big(\sum_{i=1}^n x_i^2\Big)}_{=\|x\|_2^2}$$

$$\sum c_j \cdot d_j \leq \Big(\sum c_j^2\Big)^{1/2} \Big(\sum d_j^2\Big)^{1/2}$$

so $\dfrac{\|Tx\|_2}{\|x\|_2} \leq \|A\|_{HS}$

so $\|T\| \leq \|A\|_{HS}$

$\boxed{\checkmark}$

Remark:

- In general "$<$"
- A symmetric, $\|T\| = \sup\limits_{\lambda_j \text{ eigval of } A} |\lambda_j|$.

Theorem II. 3:

Let
- $(X, \|-\|_X)$ n.s.
- $(Y, \|-\|_Y)$ Banach

Then $(L(X,Y), \|-\|_{L(X,Y)})$ is a Banach space.

Proof: Easy to check $\|-\|_{L(X,Y)}$ is a norm.

So to show completeness:

wlog $X \neq \{0\}$.

Let (T_n) C.S. in $L(X,Y)$

Then $\forall x \in X$ we have that $(T_n(x)) \subset Y$ is a Cauchy sequence as

$$\|\underbrace{T_n x - T_m x}_{(T_n - T_m)(x)}\| \leq \|T_n - T_m\| \cdot \|x\| \xrightarrow[n,m \to \infty]{} 0$$

As Y Banach $\exists\, Tx \in Y$ s.t.

$T_n x \to Tx$ in Y.

Claim: $T \in L(X,Y)$ and $\|T_n - T\| \to 0$.

Proof: Let $\varepsilon > 0$, $N \in \mathbb{N}$ s.t. $\forall n,m \geq N$ $\|T_n - T_m\| < \varepsilon$.

Then $\forall x \in X$ we have

$$\|T_n x - Tx\| = \|T_n x - \lim T_m x\|$$
$$= \lim_{m \to \infty} \underbrace{\|T_n x - T_m x\|}_{\leq \|T_n - T_m\| \cdot \|x\|}$$

$\leq \overline{\lim_{m\to\infty}} \|T_n - T_m\| \cdot \|x\|$

so for $n \geq N$

$\leq \varepsilon \cdot \|x\|$

Hence $\forall x \in X$

- $\|Tx\| \leq (\|T_N\| + \varepsilon) \cdot \|x\|$
  so $T \in L(X,Y)$
- $\|T - T_n\| \leq \varepsilon \quad \forall n \geq N$

as $\varepsilon > 0$ was arbitrary
we get $T_n \to T$ in $L(X,Y)$.

$\boxed{\text{VI}}$ Thm.

<u>Cor.:</u>
$\forall$ normed space $(X, \|\cdot\|_X)$
we have that the dualspace
$(X^*, \|\cdot\|_{X^*}) = (L(X, \mathbb{F}), \|\cdot\|_{L(X,\mathbb{F})})$
is a Banachspace.

<u>Proof</u>: $(\mathbb{F}, |\cdot|)$ is complete $\boxed{\text{VII}}$