II. Bounded linear operators (video 4)

Last time:

- Y Banach $\Rightarrow$ $L(X, Y)$ Banach

This time:

- Composition of bounded lin. op. gives a - - - - -
- Neumann series & invertibility of $Id - T$
- One more example

Recall also:

Lemma I.4

X Banach $\Leftrightarrow$ absolute convergence implies convergence

Prop. II.4:

Let X, Y, Z n.s.

and let $T \in L(X,Y)$

$S \in L(Y,Z)$.

Then $ST = S \circ T \in L(X, Z)$

and $\|ST\|_{L(X,Z)} \leq \|S\|_{L(Y,Z)} \cdot \|T\|_{L(X,Y)}$

Proof: • ST linear as S, T are linear

• $\forall x \in X$

$\|STx\|_Z = \|S(Tx)\| \leq \|S\| \cdot \|Tx\|_Y$

$\leq \|S\| \cdot \|T\| \cdot \|x\|_X$

so $ST \in L(X,Z)$, $\|ST\| \leq \|S\| \cdot \|T\|$

Application 1:

Suppose $T_n \to T$ in $L(X,Y)$

$S_n \to S$ in $L(Y,Z)$

Then $S_n T_n \to ST$ in $L(X,Z)$.

Proof:

$$\|S_nT_n - ST\| \overset{\triangle}{\leq} \|S_nT_n - S_nT\| + \|S_nT - ST\|$$

$$= \|S_n(T_n - T)\| + \|(S_n - S)T\|$$

$$\leq \underbrace{\|S_n\|}_{\leq C} \cdot \underbrace{\|T_n - T\|}_{\to 0} + \underbrace{\|S_n - S\|}_{\to 0} \cdot \underbrace{\|T\|}_{\text{fixed}}$$

$\to 0$

where $C = \sup \|S_n\| < \infty$ as convergent sequences are bounded.

Application 2: $\exp(T)$, $T \in L(X)$

Let X Banach, $T \in L(X)$

Then $\sum_{j=0}^{\infty} \frac{T^j}{j!}$ converges

in $L(X)$, define

$$\exp(T) = \sum_{j=0}^{\infty} \frac{T^j}{j!} \in L(X).$$

Proof: As X is Banach also

$L(X)$ is Banach.

Furthermore $\left\| \frac{T^j}{j!} \right\| \leq \frac{\|T\|^j}{j!}$

so as $\sum \frac{\|T\|^j}{j!} = \exp(\|T\|) < \infty$

by comparison we know that

$\sum \frac{T^j}{j!}$ converges absolutely

so $\sum \frac{T^j}{j!}$ converges in Banach

space $L(X)$. $\blacksquare$

Important application: Neumann series:

Lemma II.5 (Neumann series)

Let X Banach space, $T \in L(X)$, assume $\|T\| < 1$.

Then $Id_X - T$ is invertible, and $(Id-T)^{-1} = \sum_{j=0}^{\infty} T^j \in L(X)$.

Def.: $S \in L(X)$ is invertible if $\exists\, T \in L(X)$ s.t. $ST = TS = Id_X$.

⌈ S bijective, linear $\not\Rightarrow$ S invertible

$\Downarrow$

$\exists$ "algebraic inverse" $T: X \to X$ s.t. T linear $ST = TS = Id$

If this T is also a bounded lin. operator then S is invertible ⌋

Proof: As X Banach also $L(X)$ Banach.

As $\|T\| < 1$ we know that $\sum_{j=0}^{\infty} \|T^j\| < \infty$ since $\|T^j\| \leq \|T\|^j$.

Hence by Lemma I.4, $S_n := \sum_{j=0}^{n} T^j$ converges in $L(X)$,

say $S_n \to S$ in $L(X)$

Note: $S_n \circ (Id - T) = Id - T + T + \dots - T^{n+1}$
$= Id - T^{n+1} \to Id$

as $\|T^{n+1}\| \leq \|T\|^{n+1} \to 0$

As $S_n \to S$ also

$$S_n(Id-T) \to S(Id-T)$$

so $S(Id-T) = Id$.

Similarly $(Id-T)S = Id$

so $$\boxed{(Id-S)^{-1} = \sum_{j=0}^{\infty} T^j}$$

∎

Cor. II.6:

Let
- X Banach
- $T \in L(X)$ invertible
- $S \in L(X)$ s.t. $$\|S\| < \frac{1}{\|T^{-1}\|}$$

Then $T-S$ is invertible.

Remark:

This implies that

$$GL(X) := \{T \in L(X) : \text{invertible}\}$$

is open subset of $L(X)$.

⌜ given $T \in GL(X)$ take $\varepsilon = \frac{1}{\|T^{-1}\|}$

get $B_\varepsilon(T) \subset GL(X)$ ⌟

Proof of cor:

Use: Composition of invertible operators is invertible ($\to$ p.s. 2)

$$T - S = \underbrace{T}_{\text{inv.}}\,\underbrace{(Id - \overbrace{T^{-1}S})}_{\text{inv. by Lemma II.5}}$$

$$\|T^{-1}S\| \le \|T^{-1}\| \cdot \|S\| < 1$$

∎

Ex. 5 Integral operator on

$X = C(I, \mathbb{R})$, $I \subset \mathbb{R}$

compact interval

Given: $k : I \times I \to \mathbb{R}$ continuous

we define

$$T : f \mapsto Tf$$

$$(Tf)(t) := \int_I k(t,s) f(s) ds$$

Note:

- $Tf : I \to \mathbb{R}$ is welldefined function as $s \mapsto k(t,s) f(s)$ continuous $\forall t$ so $\int$ is well defined.

- As I compact, also $I \times I$ compact so $|k|$ is bounded on $I \times I$

$$\|k\|_{sup} = \sup_{I \times I} |k| < \infty .$$

We can bound $\forall (t,s) \in I \times I$

$$|k(t,s) f(s)| \leq \underset{\text{over } I \times I}{\|k\|_{sup}} \cdot \underset{\text{over } I}{\|f\|_{sup}}$$

and hence

$$|(Tf)(t)| \leq \underbrace{(|I| \cdot \|k\|_{sup})}_{M} \cdot \|f\|_{sup}$$

ie. Tf is bounded with

$$\|Tf\|_{sup} \leq M \cdot \|f\|_{sup}$$

⚠ Not enough to say $T \in L(X)$

Remains to show:

Claim: $\forall f \in C^0(I)$ also

$$Tf : I \to \mathbb{R} \text{ is continuous}$$

Proof: Let $t_0 \in I$, t_n any sequence in I s.t. $t_n \to t_0$

Need to show

$$(Tf)(t_n) \longrightarrow (Tf)(t_0).$$

Let $g_n(s) := k(t_n, s) \cdot f(s)$

$g(s) := k(t_0, s) f(s)$

Note: As k is continuous we have $g_n(s) \to g(s) \; \forall s \in I$

also $|g_n(s)| \leq \|k\|_{sup} \cdot \|f\|_{sup}$

$\in \mathcal{L}^1(I)$

as I bounded

so by DCT from integration we get

$$(Tf)(t_n) = \int_I g_n(s)\,ds \to \int g(s)\,ds = (Tf)(t_0)$$

$\boxed{///}$ claim.

So $T \in L(X)$

with $\|T\| \leq |I| \cdot \|k\|_{sup}$

$\boxed{///}$