Chapter III: Finite dimensional normed spaces:

Know:

Heine-Borel for $(\mathbb{F}^n, \|\cdot\|_2)$

All norms on $\mathbb{F}^n$ equivalent (Prop 1)

All norms on X, $\dim X < \infty$, are equivalent (Thm 2)

If $\{f_j\}_{j=1}^n$ basis of X then
$$Q : \mathbb{F}^n \to X, \quad \mu \mapsto \sum_{j=1}^n \mu_j f_j$$
bijection, Q, Q^{-1} linear.

All linear maps $T : (X, \|\cdot\|_X) \to (Y, \|\cdot\|_Y)$, $\dim X < \infty$ are continuous (Thm 3)

$(\mathbb{F}^n, \|\cdot\|_2)$ complete

$\dim X < \infty \iff$ Heine Borel holds (Thm 7)

$\dim X = n \Rightarrow X$ homeomorphic to $\mathbb{F}^n$ (Cor 4)

Any $(X, \|\cdot\|_X)$, $\dim X < \infty$ is complete (Thm 5)

Riesz-Lemma (Lemma 8)

Any finite dim. subspace is closed

Prop. III.1:

On $\mathbb{F}^n$, $n \in \mathbb{N}$, all norms are equivalent to $\|\cdot\|_2$, so in particular any two norms $\|\cdot\|$, $\|\cdot\|'$ are equivalent.

Proof: Let $\|\cdot\|$ be any norm on $\mathbb{F}^n$

Claim 1: $\exists\, C_1$ s.t. $\forall x \in \mathbb{F}^n$

$$\|x\| \leq C_1 \cdot \|x\|_2.$$

Proof:

Let $e_1, \dots, e_n$ standard basis of $\mathbb{F}^n$,

set $C_1 = \left(\sum_{j=1}^n \|e_j\|^2 \right)^{1/2}$

Then $\forall x \in \mathbb{F}^n$

$$\|x\| = \left\| \sum_{j=1}^n x_j e_j \right\| \overset{\triangle}{\leq} \sum_{j=1}^n |x_j| \cdot \|e_j\|$$

$$\underset{C-S}{\leq} \underbrace{\left(\sum_{j=1}^n |x_j|^2 \right)^{1/2}}_{= \|x\|_2} \cdot \underbrace{\left(\sum_{j=1}^n \|e_j\|^2 \right)^{1/2}}_{= C_1}$$

$\boxed{}$ claim 1.

Claim 2: $\exists\, C_2$ s.t.

$$\|x\|_2 \leq C_2 \|x\| \quad \forall x \in X.$$

Proof vers 1:

Let

$$f: (\mathbb{F}^n, \|-\|_2) \to \mathbb{R}$$
$$x \mapsto f(x) = \|x\|$$

Note that

$$|f(x) - f(y)| = |\|x\| - \|y\||$$
$$\leq \|x-y\|$$
$$\overset{\text{claim 1}}{\leq} C_1 \cdot \|x-y\|_2$$

so f is Lipschitz (const $\leq C_1$)

so f continuous.

Know that $S = \{x : \|x\|_2 = 1\}$

bounded and closed in $(\mathbb{F}^n, \|-\|_2)$

so ~~B~~ Heine-Borel compact.

So

$$\min_{x \in S} f(x) = f(x_0) = \|x_0\| > 0$$

is achieved, say at some $x_0 \in S$

Set $C_2 = \frac{1}{\|x_0\|}$.

Note if $x = 0$ clearly $\|x\|_2 = 0 \leq 0 = C_2\|x\|$

while for $x \neq 0$

we consider $\tilde{x} = \frac{x}{\|x\|_2} \in S$ so

$$\|\tilde{x}\| \geq \|x_0\|$$

i.e. $\frac{\|x\|}{\|x\|_2} \geq \|x_0\| \iff \|x\|_2 \leq \underbrace{\frac{1}{\|x_0\|}}_{C_2} \|x\|$

$\boxed{}$ claim 2

$\boxed{}$ Thm. 1.

Proof of claim 2, variant 2:

If claim 2 is wrong, then

$\forall n \in \mathbb{N} \;\; \exists\, x_n \in \mathbb{F}^n$ s.t.

$$\|x_n\|_2 > n \cdot \|x_n\| \qquad (\Rightarrow x_n \neq 0)$$

Let $\tilde{x}_n = \dfrac{x_n}{\|x_n\|_2}$

note that
- $\|\tilde{x}_n\|_2 = 1$ so $\tilde{x}_n \in S$
- $\|\tilde{x}_n\| < \frac{1}{n} \to 0$

so $\tilde{x}_n \to 0$ wrt $\|-\|$

As $S \subset (\mathbb{F}^n, \|-\|_2)$ is compact, $\exists$ subseq. (x_{n_j}) s.t. $x_{n_j} \xrightarrow{\|-\|_2} x$, some $x \in S$.

Obviously $x \neq 0$ since $\|x\|_2$

so also $\|x\| \neq 0$.

But at the same time

$$\|\tilde{x}_n - x\| \underset{\uparrow \text{ claim 1}}{\leq} C_1 \cdot \|\tilde{x}_n - x\|_2 \longrightarrow 0$$

so $\tilde{x}_n \xrightarrow{\|-\|} x \neq 0$ ↯ to $\tilde{x}_n \xrightarrow{\|-\|} 0$

∎

Thm. III.2

Let X be any finite dim. v.s.

Then any two norms on X are equivalent.

<u>Proof</u>: Let $\|\cdot\|, \|\cdot\|'$ be norms on X, let $Q: \mathbb{F}^n \to X$ as above.

Set $\|\mu\|_{\mathbb{F}^n} := \|Q\mu\|$

$\|\mu\|'_{\mathbb{F}^n} := \|Q\mu\|'$

These are norms on $\mathbb{F}^n$ since

- ≥ 0 ✓, definite as Q injective
- Δ, (N2) from linearity of Q & corresp. prop. of $\|\cdot\|, \|\cdot\|'$

So Prop 1 gives $\exists C > 0$ s.t.

$$\frac{1}{C}\|\mu\|_{\mathbb{F}^n} \leq \|\mu\|'_{\mathbb{F}^n} \leq C \cdot \|\mu\|_{\mathbb{F}^n}$$

As Q is surjective can write any $x \in X$ as $Q\mu$ and use above to get

$$\ldots \leq \|x\| = \|\mu\|_{\mathbb{F}^n} \leq C \cdot \|\mu\|'_{\mathbb{F}^n} = C \cdot \|x\|'$$

$\boxed{\text{VII}}$

Theorem III.3:

Let $(X, \|-\|_X)$ is a finite dim. normed space and $(Y, \|-\|_Y)$ any normed space.

Then any linear map $T: X \to Y$ is continuous.

Proof: Define

$$\|x\| := \|x\|_X + \|Tx\|_Y$$

Easy to check: Is a norm.

So Thm 2 gives $\exists C$ s.t.

$$\|Tx\|_Y \leq \|x\| \leq C \cdot \|x\|_X$$

$\blacksquare$

Cor. III.4:

Let $(X, \|-\|_X)$ be any finite dim. n.s. Then $(X, \|-\|_X)$ is homeomorphic to $(\mathbb{F}^n, \|-\|_2)$, $n = \dim X$.

Proof: Know that $Q: \mathbb{F}^n \to X$ linear, bijective so $Q^{-1}: X \to \mathbb{F}^n$ linear.

As both $\dim \mathbb{F}^n, \dim X < \infty$ Thm 3 implies that Q, Q^{-1} are continuous (indeed Lipschitz!)

Thm. III.5

Every finite dimensional normed space $(X, \|\cdot\|_X)$ is a Banach space.

Proof: Let (x_n) C.S. in $(X, \|\cdot\|)$

Let $Q : \mathbb{F}^n \to X$

As $Q^{-1} : X \to \mathbb{F}^n$ is Lipschitz we have that $(Q^{-1}x_n)$ is C.S. in $(\mathbb{F}^n, \|\cdot\|_2)$

indeed

$$\|Q^{-1}x_n - Q^{-1}x_m\|_2 = \|Q^{-1}(x_n - x_m)\|$$
$$\leq \|Q^{-1}\| \cdot \|x_n - x_m\|$$
$$\to 0.$$

As $(\mathbb{F}^n, \|\cdot\|_2)$ is complete $\exists\, y \in \mathbb{F}^n$ s.t. $Q^{-1}x_n \to y$.

Set $x = Qy$.

$$\|x - x_n\| = \|Qy - QQ^{-1}x_n\|$$
$$\leq \|Q\| \, \|y - Q^{-1}x_n\|$$
$$\to 0$$

as Q is a bounded lin op.

so $x_n \to x$ in $(X, \|\cdot\|)$ $\blacksquare$

Cor. III.6:

Any finite dim. subspace Y of any normed space $(X, \|\cdot\|_X)$ is closed

Proof: $(Y, \|\cdot\|_X)$ is complete $\to$ Lemma 1.1 + foll. remark $\blacksquare$