Thm. III.7:

Let $(X, \|\cdot\|)$ n.s. Then the following are equivalent:

(i) $\dim X < \infty$

(ii) Every bounded and closed subset is compact.

(iii) $S = \{x \in X : \|x\| = 1\}$ compact.

Need

Prop. III.8 (Riesz-Lemma):

Let
- $(X, \|\cdot\|)$ any normed space
- $Y \subsetneq X$ subspace, closed

Then $\forall \varepsilon > 0 \ \exists x \in X$ with $\|x\| = 1$

s.t. $\operatorname{dist}(x, Y) = \inf\{\|x-y\| : y \in Y\}$

$\geq 1 - \varepsilon$.

Remark:

If $(X, \|\cdot\|)$ is a Hilbert space, say for some inner product $(-,-)$, then $\forall Y \subsetneq X$ closed subspace $\exists x \in X$ s.t. $\operatorname{dist}(x, Y) = 1$, $\|x\| = 1$.

Reason: If Y closed then

$$X = Y \oplus Y^\perp, \quad Y^\perp = \{x \in X : (x, y) = 0\}$$

any element $x \in Y^\perp$ will have $\forall y \in Y$ $\|x-y\|^2 = \|x\|^2 - 2\underbrace{(x,y)}_{=0} + \|y\|^2$

$= \|x\|^2 + \|y\|^2 \geq \|x\|^2$

Proof of Prop III.8:

Let $x^* \in X \setminus Y$ (as $Y \subsetneq X$)

as Y is closed, $X \setminus Y$ is open

so $\exists r > 0$ with

$$B_r(x^*) \subset X \setminus Y.$$

In particular

$$d := \operatorname{dist}(x^*, Y) \geq r > 0$$
$$\parallel$$
$$\inf\{\|x^* - y\| : y \in Y\}.$$

...

Let $\varepsilon > 0$ note that since $d > 0$

we have $\frac{d}{1-\varepsilon} > d$, wlog $\varepsilon \in (0,1)$.

so $\exists$ ~~...~~ $y^* \in Y$ s.t.

$$\|x^* - y^*\| \leq \frac{d}{1-\varepsilon}$$

Claim: For $x := \frac{x^* - y^*}{\|x^* - y^*\|} \in S$

we have

$$\operatorname{dist}(x, Y) \geq 1 - \varepsilon.$$

Proof: Let $y \in Y$, also $\tilde{y} = \|x^* - y^*\| \cdot y \in Y$

$$\|x - y\| = \left\| \frac{x^* \underbrace{- y^* - \tilde{y}}_{= -(y^* + \tilde{y}) \in Y}}{\|x^* - y^*\|} \right\|$$

$$= \underbrace{\frac{1}{\|x^* - y^*\|}}_{\leq \frac{1-\varepsilon}{d}} \cdot \underbrace{\|x^* - \tilde{\tilde{y}}\|}_{\geq d}$$

with $\tilde{\tilde{y}} \in Y$

$$\leq 1 - \varepsilon$$

$\boxed{\text{k}}$

Proof of Thm III.7:

(i) => (ii)

Let $Q: (\mathbb{F}^n, \|\cdot\|_2) \to (X, \|\cdot\|)$

bilipschitz map from above.

Given any $F \subset X$ which is

- bounded, i.e. $\exists R > 0$ s.t.
  $F \subset \overline{B_R(0)}$

- closed.

Then $Q^{-1}(F)$ is

- bounded as $Q^{-1}(F) \subset \overline{B_{R\|Q^{-1}\|}(0)} \subset \mathbb{F}^n$

since $\|Q^{-1}x\| \leq \|Q^{-1}\| \cdot \underbrace{\|x\|}_{\leq R}$
$\forall x \in F$

- $Q^{-1}F$ = preimage of closed set F under the continuous map Q

  so closed.

Hence by H-B on $(\mathbb{F}^n, \|\cdot\|_2)$ we have that

$Q^{-1}F$ is compact.

So $F = Q(Q^{-1}F)$

is image of a compact set under a continuous Q

$\to$ compact.

(ii) => (iii) as S bounded & closed

$S = f^{-1}(\{1\})$
$f(x) = \|x\|$

(iii) => (i)

Suppose wrong, ie.

$\exists\ (X, \|\cdot\|_X)$ with • S compact

• $\dim X = \infty$.

Let $\{y_1, y_2, \dots\}$ a set of linearly independent elements of X.

and let

$Y_k := \operatorname{span}\{y_1, \dots, y_k\}$

Then $Y_k \subsetneq Y_{k+1}$

and since Y_k is finite dim.

we have that Y_k is closed subspace of $(Y_{k+1}, \|\cdot\|_X)$.

Riesz lemma gives $\rightarrow y_k \in S$

$\exists\ y_k \in Y_{k+1},\ \|y_k\| = 1$

s.t. $\operatorname{dist}(y_k, Y_k) \geq \frac{1}{2}$.

Note that if $k > \ell$ then

$k \geq \ell + 1$ so $Y_{\ell+1} \subseteq Y_k$

$\begin{array}{c}\cup\\ y_\ell\end{array}$

so

$\|y_k - y_\ell\| \geq \operatorname{dist}(y_k, Y_k) \geq \frac{1}{2}$.

Hence the sequence (y_k) cannot have any subsequence (y_{k_j}) which is Cauchy (because $\|y_{k_j} - y_{k_m}\| \geq \frac{1}{2}\ \forall j, m$)

so $(y_k) \subset S$ has no conv. subseq. $\lightning$ S compact

$\square$