IV.2 Theorem of Stone-Weierstrass

Goal: Find "good" sufficient conditions for a subspace Y of $C(K) = C(K, \mathbb{R})$, with $\|\cdot\|_{\sup}$, $K \subset \mathbb{R}^n$ compact, to be dense.

⚠ $\mathbb{F} = \mathbb{R}$

⌜Everything wrong / won't make sense in $\mathbb{F} = \mathbb{C}$⌟

Strategy:

(1) Identify a NECESSARY condition for a subspace to be dense
"Y separates point"

(2) Q: Sufficient?
A: NO, can't even approximate constant functions in general
→ ask additionally:
Y contains constants.

Q: Sufficient?
A: No

(3) Identify extra condition: Y has more "structure" than just a vector space.

TODAY:
Extra structure

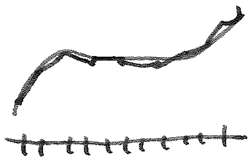
Y "sublattice"

$\downarrow$

Thm IV.3:
Sublattice version of SW

$\rightsquigarrow$

Piecewise lin. functions are dense



$\xRightarrow{\text{+work}}$

NEXT VIDEO

Y "subalgebra"
$(f, g \in Y \Rightarrow f \cdot g \in Y)$

$\downarrow$

Thm IV.6
Subalgebra version of SW

$\downarrow$

Polynomials are dense $C(K)$

(Weierstrass $K = I \subset \mathbb{R}$

St.-W. in $K \subset \mathbb{R}^n$)

Identifying a necessary condition:

Given any $p \neq q$, $p, q \in K$

$\exists f \in C(K)$ s.t. $f(p) \neq f(q)$.

$\ulcorner$ E.g. $f(x) = |x-p|^2 \dots$ $\lrcorner$

Note if $Y \subset C(K)$ is dense then there must also be some $g \in C(K)$ s.t. $g(p) \neq g(q)$.

Why? If I had that $g(p) = g(q)$ $\forall g \in Y$ then

$$Y \subset \tilde{Y} := \{g \in C(K) : g(p) = g(q)\}$$

closed

so $\overline{Y} \subset \overline{(\tilde{Y})} = \tilde{Y} \subsetneq C(K)$

Def. 9: A subspace $L \subset C(K)$ is called a sublattice if

$f, g \in L \Rightarrow \max(f,g) \in L.$

Note: For v.s. L

L sublattice $\Leftrightarrow (f,g \in L \Rightarrow \min(f,g) \in L)$

$\Leftrightarrow (f \in L \Rightarrow |f| \in L)$

$\ulcorner -\min(-x,-y) = \max(x,y)$

$|x| = \max(x,0)$, $\max(x,y) = \frac{1}{2}(x+y) + \frac{1}{2}|x-y|$ $\lrcorner$

Thm. IV.3 (Stone-Weierstrass, sublattice version)

Let $K \subset \mathbb{R}^n$ compact, $(C(K), \|\cdot\|_{\sup})$

Suppose: $L \subset C(K)$ subspace s.t.

(i) L separates points

(ii) L contains the const. functions

(iii) L is a sublattice

Then $L \subset C(K)$ is dense.

Remark: E.g. piecewise linear functions are dense in $C(I)$ as span$\{1, x\}$ (i), (ii) ✓ & sublattice.

Note: (i) & (iii) $\Rightarrow$ dense:

Take $L = \{f \in C(K) : f(p) = 0\}$

for some $p \in K$. Satisfies (i), (iii)

Not dense as L closed

$\overline{L} = L \neq C(K)$ $\square$

Strategy for proof:

Let $f \in C(K)$, $\varepsilon > 0$.

1. Show that $\forall p, q \in K$
$\exists f_{p,q} \in L$ s.t.
$f_{p,q}(p) = f(p)$ and $f_{p,q}(q) = f(q)$.

take suitable min

2. Show that $\forall p \in K$
$\exists g_p \in L$ s.t. $g_p(p) = f(p)$
and $g_p < f + \varepsilon$ on K

suit. max of g_p's

3. $\exists g \in L$ s.t. $f - \varepsilon < g < f + \varepsilon$ on K.

Lemma IV.4: ($\hat{=}$ Step 1)

Let $L \subset C(K)$ subspace s.t.

- contains constants
- separates points

Then $\forall f \in C(K)$, $\forall p, q \in K$ we have that $\exists f_{p,q} \in L$ s.t.

$$f_{p,q}(p) = f(p), \quad f_{p,q}(q) = f(q).$$

Furthermore $\forall \varepsilon > 0$ $\exists$ open subset $U_\varepsilon^{p,q}$ of K s.t. on U_ε we have

$$|f_{p,q} - f| < \varepsilon,$$

and $p, q \in U_\varepsilon^{p,q}$.

Proof:

- $p = q \rightarrow f_{p,q}(x) = f(p) \ \forall x \in K$
- $p \neq q$ $\exists g \in L$ s.t.

$$g(p) = 1, \quad g(q) = 0$$

Take $f_{p,q}(x) = f(q) + g(x) \cdot (f(p) - f(q))$

$\in L$

Note as $f_{p,q} - f$ is continuous we have $(f_{p,q} - f)^{-1}((-\varepsilon, \varepsilon)) =: U_\varepsilon^{p,q}$ is open in K and contains p, q.

Step 2: Let $p \in K$ fixed.

As $q \in U_\varepsilon^{p,q}$ $\forall q \in K$

we have that $K = \bigcup_{q \in K} U_\varepsilon^{p,q}$

so as $U_\varepsilon^{p,q}$ are open we have

an open cover of compact set K.

Thus $\exists q_1, \dots, q_m$ s.t.

$$K = \bigcup_{j=1}^{m} U_\varepsilon^{p,q_j}.$$

We set

$$g_p := \min\{f_{p,q_1}, \dots, f_{p,q_m}\} \in L.$$

Note that as $f_{p,q}(p) = f(p)$ $\forall q$

we have

$g_p(p) = f(p)$.

Also $\forall x \in K$ $\exists j$ s.t. $x \in U_\varepsilon^{p,q_j}$

so

$g_p(x) \leq f_{p,q_j}(x) < f + \varepsilon$

so $g_p < f + \varepsilon$ on all of K.

Step 3:

Let $V_p := (f - g_p)^{-1}((-\varepsilon, \varepsilon))$

open and note that $p \in V_p$

as $(f - g_p)(p) = 0$.

So $(V_p)_{p \in K}$ is open cover of K

$\exists p_1, \dots, p_\ell \in K$ s.t. $\bigcup_{j=1}^{\ell} V_{p_j} = K$.

Set $g = \max\{g_{p_1}, \dots, g_{p_\ell}\}$.

Then

- $g < f+\varepsilon$ on K as all
  $g_p < f+\varepsilon$ on K.

- $\forall x \in K$ let j s.t. $x \in V_{p_j}$
  then
  $g(x) \geq g_{p_j}(x) > f-\varepsilon$

so $g > f-\varepsilon$ on K

i.e. $g \in L$
$\|f-g\|_{sup} \leq \varepsilon$ on K

Holds $\forall f \in C(K)$ $\forall \varepsilon > 0$

$\boxed{///}$ Thm of SW
lattice vers.