IV.2 Theorem of Stone-Weierstrass (part 2)

$C(K) = C(K, \mathbb{R})$, $K \subset \mathbb{R}^n$ compact

Last time:

Sublatticeversion of SW $\Longrightarrow$ Thm IV.6 subalgebra vers. of SW

\+

prop. IV.7: Closed subalg is a sublattice $\Longrightarrow$ Thm IV.6

Thm IV.6 $\Longrightarrow$ polynomials are dense

polynomials are dense $\Longrightarrow$ (next chapter) C(K) contains a dense countable subset i.e. C(K) is "separable"

Thm. IV.5 (Thm of Weierstrass 1885 (n=1)

Thm of Stone-Weierstrass 1937)

Let $K \subset \mathbb{R}^n$ compact. Then the space of polynomials is dense in $C(K)$.

Def. 10: A <u>subspace</u> $A \subset C(K)$ is called a subalgebra if

- $f \equiv 1$ is in A (e.g. all const. fn.)
- $f, g \in A \Rightarrow f \cdot g \in A$.

Thm IV.6 (Subalgebra vers. of SW)

Let $K \subset \mathbb{R}^n$ compact.

Let $A \subset C(K)$ is subalgebra which separates points.

Then $A \subset C(K)$ is dense.

To prove this all we have to show

Prop. IV. 7:

If $A \subset C(K)$ subalgebra & closed then A is a sublattice.

Note Prop. 7 + Sublattice vers. of SW

$\implies$ Subalg vers.

Since given a subalg. A we get that also $\bar{A}$ is a subalg since
- $1 \in A \subset \bar{A}$
- $f, g \in \bar{A}$, $\exists f_n \in A, g_n \in A$ s.t. $f_n \to f$, $g_n \to g$

$$\underbrace{f_n \cdot g_n}_{\in A} \to f \cdot g \text{ so } f \cdot g \in \bar{A}$$

So assumpt. of Thm 6 + Prop 7 give that $\bar{A}$ is sublattice which sep. points, so by Thm. 4

$$\underset{\bar{A}}{\overset{\|}{\overline{(\bar{A})}}} = C(K) \quad \to A \subset C(K) \text{ dense.}$$

For proof of Prop. 7 we use

Lemma IV. 8 (General version of Dini's lemma for compact metric spaces)

Let

- (M, d) compact metric space
- seq. of continuous functions $g_n : M \to \mathbb{R}$

s.t.
- $g_n(x) \to 0 \quad \forall x \in M$
- $g_{n+1}(x) \leq g_n(x) \quad \forall x \in M \; \forall n$

Then $g_n \to 0$ uniformly on M.

Proof of lemma 8:

Let $\varepsilon > 0$

Consider

$$U_n := \{x \in M : g_n(x) < \varepsilon\}$$
$$= g_n^{-1}((-\varepsilon, \varepsilon)) \quad \text{open}$$

As $g_{n+1} \leq g_n$ also

$$U_{n+1} \supseteq U_n$$

and as $g_n \to 0$ pointwise : $\bigcup_{n=1}^{\infty} U_n = M$

So by compactness of M $\exists$ finite subcover

i.e. $\exists N$ s.t.

$$U_N = \bigcup_{n=1}^{N} U_n = M$$

so $\forall n \geq N$

$$0 \leq g_n(x) \leq g_N(x) < \varepsilon$$

$\forall x \in M$

$\boxed{\text{IV}}$ Lemma 8

Proof of Prop. 7:

To show: If $f \in A$ then $|f| \in A$.

Case 1: $\|f\|_{sup} \leq 1$.

Let $f_0 \equiv 0 \in A$

$$f_{n+1} = f_n + \frac{1}{2}(f^2 - f_n^2) \in A$$

Claim: $\forall n \in \mathbb{N}_0$

$$0 \leq f_n \leq |f|$$

and $f_{n+1} \geq f_n$.

Proof by induction:

- $n=0$ $\quad f_0 = 0 \leq f_1 = \frac{1}{2}f^2$

- Assume true for n

Then

- $f_{n+1} \geq f_n \geq 0$

- $f_{n+1} = f_n + \frac{1}{2}\underbrace{(|f| + |f_n|)}_{\leq 2|f| \leq 2} \cdot \underbrace{(|f| - f_n)}_{\geq 0}$

$$\leq f_n + |f| - f_n = |f|$$

- $f_{n+2} - f_{n+1} = f_{n+1} - f_n$

$$- \frac{1}{2}\underbrace{(f_{n+1}^2 - f_n^2)}_{= \underbrace{(f_{n+1}+f_n)}_{\leq 2} \cdot \underbrace{(f_{n+1}-f_n)}_{\geq 0}}$$

$$\geq f_{n+1} - f_n - \frac{1}{2} \cdot 2 \cdot (f_{n+1} - f_n)$$

$$= 0.$$

$\boxed{//}$ claim.

Hence $\forall x \in K$

$(f_n(x))$ bounded above, $\nearrow$ in $\mathbb{R}$

so converges to some $h(x) \geq 0$

By AOL

$h(x) = \lim f_{n+1}(x)$

$= h(x) + \frac{1}{2}(f^2(x) - h^2(x))$

so $h^2(x) = f^2(x)$

so $h(x) = |f(x)|$ since $h(x) \geq 0$.

Thus : $f_n \longrightarrow h = |f|$ pointwise

Note: $g_n = |f| - f_n \to 0$

pointwise and is monotone

$g_{n+1} \leq g_n$

so by Dini's Lemma $g_n \to 0$

uniformly, i.e. $f_n \to |f|$ uniformly.

So $|f| \in A$ since
- $f_n \in A$
- A closed.

Case 2:

If $f \in A$, $\|f\|_{sup} > 1$

Then $\tilde{f} = \frac{f}{\|f\|_{sup}} \in A$, $\|\tilde{f}\|_{sup} \leq 1$

so $|\tilde{f}| \in A$ by case 1

so $|f| = \|f\|_{sup} \cdot |\tilde{f}| \in A$ VII prop. 7.

Application:

Let $f \in C[0,1] = X$ s.t.

$$\int_0^1 f(t)\, t^n \, dt = 0 \qquad \forall n \in \mathbb{N}_0 .$$

Then f needs to be $f \equiv 0$.

Proof:

Let $T : X \to \mathbb{R}$

$$T(g) := \int_0^1 f(t) g(t)\, dt.$$

- T linear
- $|T(g)| \le \|f\|_{sup} \cdot \|g\|_{sup}$

so $T \in L(X, \mathbb{R}) = X^*$

As $T(t^n) = 0 \ \forall n \in \mathbb{N}_0$

we have

$$T(p) = 0 \quad \forall p \text{ polynomial.}$$

Thus $T|_{\{\text{polynomials}\}} = 0$

so as $\{\text{polyn.}\} \subset X$ is dense

get $T \equiv 0$ by Lemma 4.2.

In particular

$$T(f) = 0, \qquad T(f) = \int_0^1 f^2(t)\, dt$$

so $f(t) = 0$ a.e $t \in [0,1]$.

As f continuous hence $f(t) = 0 \ \forall t \in [0,1]$.

$\blacksquare$