Chapter V:

Separability

<u>Def 11:</u> A normed space $(X, \|-\|)$ is separable if $\exists\, D \subset X$ which is dense and countable.

<u>Lemma V.1:</u>

(i) If $\|-\|$, $\|-\|'$ on a space X are equivalent then $(X, \|-\|)$ separable $\Leftrightarrow$ $(X, \|-\|')$ is sep.

(ii) If $(X, \|-\|_X)$ and $(Y, \|-\|_Y)$ are isometrically isomorphic, i.e. $\exists\, \iota : X \to Y$ bijective, linear and

$$\|\iota(x)\|_Y = \|x\|_X \quad \forall x \in X$$

(and hence $\|\iota^{-1}(y)\|_X = \|y\|_Y$ $y \in Y$)

then $(Y, \|-\|_Y)$ sep. $\Leftrightarrow$ $(X, \|-\|_X)$ sep.

Prop. V.2:

Every finite dimensional normed space $(X, \|\cdot\|_X)$ is separable.

Proof: ($\mathbb{F} = \mathbb{R}$, else replace every $\mathbb{Q}$ with $\mathbb{Q} + i\mathbb{Q}$)

① $(\mathbb{R}^n, \|\cdot\|_1)$ is separable.

Let $x \in \mathbb{R}^n$, $\varepsilon > 0$.

Then as $\mathbb{Q} \subset \mathbb{R}$ are dense

$\exists\, q_i$, $i = 1, \dots, n$ s.t. $|q_i - x_i| < \frac{\varepsilon}{n}$

so $\|q - x\|_1 < \varepsilon$ ▯

② $(\mathbb{R}^n, \|\cdot\|)$ is separable by Lemma V.1 & as $\|\cdot\|$ is equiv. to $\|\cdot\|_1$.

③ Let $(X, \|\cdot\|_X)$ any finite dim. normed space.

Let $Q : \mathbb{R}^n \to X$ bounded linear, bijection, Q^{-1} - - - -

Set $\|x\|_Q = \|Qx\|_X$ $\quad x \in \mathbb{R}^n$

This is a norm on $\mathbb{R}^n$ and

$Q : (\mathbb{R}^n, \|\cdot\|_Q) \to (X, \|\cdot\|_X)$

is isometric isomorphism.

Hence by ② and Lemma V.1

$\to (X, \|\cdot\|_X)$ sep.

▯

Lemma V.4:

Let $(X, \|\cdot\|_X)$ n.s., $Y \subset X$ subspace and $D \subset Y$.

Then if $D \subset (Y, \|\cdot\|_X)$ dense

and $Y \subset (X, \|\cdot\|_X)$ dense

then $D \subset (X, \|\cdot\|_X)$ dense

Proof: $\forall x \in X \; \forall \varepsilon > 0 \; \exists y \in Y \; \|x-y\|_X = \frac{\varepsilon}{2}$

$\exists d \in D \; \|d-y\|_X < \frac{\varepsilon}{2}$

$\to \square$

Lemma V.5:

Let $(X, \|\cdot\|_X)$ normed space, suppose $\exists$ coutable set S s.t. $\operatorname{span}(\overline{S})$ is dense in X. Then $(X, \|\cdot\|_X)$ is separable.

Remark: • $\operatorname{span}(A) = \{$finite linear combinations of elements of $A\}$

• Proof for $\mathbb{F} = \mathbb{R}$

If $\mathbb{F} = \mathbb{C}$ $\to$ replace $\mathbb{Q}$ by $\mathbb{Q} + i\mathbb{Q}$.

Prop. V. 3:

$\ell^\infty(\mathbb{F})$ and $L^\infty(\Omega)$, $\Omega \subset \mathbb{R}^n$, $\mathring{\Omega} \neq \emptyset$

are not separable.

Proof:

As $\mathring{\Omega} \neq \emptyset$ $\exists\, x \in \mathring{\Omega}$ so $\exists\, \varepsilon > 0$

s.t. $B_\varepsilon(x) \subset \Omega$.

Let $A = \{ \mathbb{1}_{B_a(x)},\ a \in (0, \varepsilon] \}$

Then if $f, g \in A$, $f \neq g$

then $\| f - g \|_{L^\infty} = 1$.

This implies that $L^\infty(\Omega)$ is not separable because:

Suppose that $D \subset L^\infty(\Omega)$ is dense

Then $\forall f \in A$ $\exists\, d_f \in D$ s.t.

$$\| f - d_f \|_{L^\infty} < \frac{1}{2}$$

Then as
$$1 \leq \underbrace{\| f - g \|}_{< \frac{1}{2}} \leq \| d_f - f \| + \| d_f - d_g \| + \underbrace{\| d_g - g \|}_{< 1/2} < 1 + \| d_f - d_g \|$$

get that $\| d_f - d_g \| > 0$ so $d_f \neq d_g$ $\in D$ i.e. $A \ni f \mapsto d_f$ is injective.

So as A is uncountable also D is uncountable $\boxed{\text{VA}}$

Proof of Lemma 5:

Let $x \in X$, $\varepsilon > 0$.

As $\operatorname{span}(\bar{S}) \subset X$ dense

$\exists\ N \in \mathbb{N}$, $\bar{s_j} \in \bar{S}$, $a_j \in \mathbb{R}$
$j = 1, \dots, N$

s.t.
$$\| x - \sum_{j=1}^{N} a_j \bar{s_j} \| < \varepsilon/3 .$$

As $S \subset \bar{S}$ dense

$\exists\ s_j \in S$ s.t.
$$|a_j| \cdot \| s_j - \bar{s_j} \| < \varepsilon/3N$$

As $\mathbb{Q} \subset \mathbb{R}$ dense $\exists\ b_j \in \mathbb{Q}$ s.t.
$$|a_j - b_j| \cdot \| s_j \| < \varepsilon/3N$$

So
$$\|x - \sum_{j=1}^{N} b_j s_j \| \overset{\Delta}{\leq} \|x - \sum a_j \bar{s_j} \| + \sum_{j=1}^{N} |a_j| \cdot \| s_j - \bar{s_j} \| + \sum_{j=1}^{N} |a_j - b_j| \cdot \| s_j \|$$
$$< \varepsilon$$

So know that
$$D = \{ \sum_{j=1}^{N} b_j s_j ,\ s_j \in S,\ b_j \in \mathbb{Q},\ N \in \mathbb{N} \}$$

$\subset X$ is dense.

Claim: D countable

Proof: As S is countable, can write $S = \{s_1, s_2, \dots\}$

Consider

$$A = \bigcup_{N \in \mathbb{N}} \overbrace{\{ (b_1, b_2, \dots,) \text{ s.t. } b_j \in \mathbb{Q} \; \forall j, \; b_j = 0 \; \forall j \geq N \}}^{A_N}$$

$$\Big\downarrow \varphi$$

$$D$$

$$\varphi((b_1, \dots,)) = \sum_{j=1}^{\dots} b_j s_j$$

is surjective so as

A = coutable union of sets A_N

where $A_N \cong \mathbb{Q}^N$ ← count.

so A countable

we get tha D countable and hence X is separable $\boxed{///}$

Prop. V.6:

(i) $(C(K), \|\cdot\|_{sup})$ is separable $\forall K \subset \mathbb{R}^n$

(ii) $\ell^p(\mathbb{F})$, $1 \leq p < \infty$ is separable

(iii) $L^p(K)$ is separable $\forall K \subset \mathbb{R}^n$ compact.

Proof:

(i) $\mathcal{P}(K) = \text{span}\{ x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}, \alpha \in \mathbb{N}_0^n \}$

$\subset C(K)$ dense

$\to$ follows from lemma since $\mathbb{N}_0^n$ is countable

(ii) Let

$S = \{ e^{(j)} = (0, \dots, 0, 1, 0, \dots) \}$

countable

and $\text{span}(S) \subset \ell^p(\mathbb{F})$ dense

since for $x \in \ell^p(\mathbb{F})$ we have

$\sum_{j=1}^\infty |x_j|^p < \infty$

so $\sum_{j=N+1}^\infty |x_j|^p \xrightarrow[N \to \infty]{} 0$

$\| x - \sum_{j=1}^N x_j e^{(j)} \|_{\ell^p}^p$.

(iii) Variant 1 (using SW)

$\mathcal{P}(K) \subset L^p(K)$ dense

$\to$ done.

Variant 2 (using density of step fn)

For $K = [a,b]$

Know ($\to$ integration)

Step functions $= \operatorname{span}\{ \mathbb{1}_{[a,b]},\ [a,b] \subset [c,d] \}$

are dense in $L^p(K)$.

As for $S := \{ \mathbb{1}_{[\hat{a},\hat{b}]},\ \hat{a},\hat{b} \in \mathbb{Q},\ [\hat{a},\hat{b}] \subset [c,d] \}$

we have

$$\overline{S} = \{ \mathbb{1}_{[a,b]},\ [a,b] \subset [c,d] \}$$

(check!) we get that

L^p is separable again

Lemma 5. $\square$

Remark:

For $\mathbb{C}$-valued functions we can still argue that $L^p(K,\mathbb{C})$ is separable using SW provided we say that

$$\mathcal{P}(K,\mathbb{R}) \oplus i\,\mathcal{P}(K,\mathbb{R}) \subset L^p(K,\mathbb{C})$$

is dense.

Prop. V.7:

Let $(X, \|\cdot\|_X)$ be a separable normed space, let Y any subspace.

Then $(Y, \|\cdot\|_X)$ is again separable.

Proof: Let $D_X = \{x_k, k \in \mathbb{N}\} \subset X$ dense.

Then $\forall k \in \mathbb{N}$ $\forall n \in \mathbb{N}$

$\exists y_{k,n} \in Y$ s.t.

$$\|x_k - y_{k,n}\| \leq \text{dist}(x_k, Y) + \frac{1}{n}$$

Claim: $D_Y = \{y_{k,n}, k,n \in \mathbb{N}\}$ is dense in Y.

Proof:

Let $y \in Y$, $\varepsilon > 0$

As $y \in X$ $\exists x_k \in D_X$ s.t.

$$\|y - x_k\| < \frac{\varepsilon}{3}.$$

In particular $\text{dist}(x_k, Y) < \varepsilon/3$

so taking n s.t. $\frac{1}{n} < \varepsilon/3$

we get

$$\|y_{n,k} - x_k\| \leq \text{dist}(x_k, Y) + \frac{1}{n} < \frac{2\varepsilon}{3}$$

giving

$$\|y_{n,k} - y\| < \varepsilon \qquad ▨$$

Remark:

If $(X, \|\cdot\|_X)$ is separable

$\exists$ finite dim. spaces

$Y_1 \subseteq Y_2 \subseteq \dots$

s.t. $X \supset \bigcup_{n=1}^{\infty} Y_n$

Proof: $D = \{d_1, d_2, \dots\}$

$Y_k = \operatorname{span}\{d_1, \dots, d_k\}$ ☐

"Galerkin's method"

To prove/solve a problem on X, try to

① solve a approximate problem on the finite dim. spaces Y_k
→ solution y_k

② Have/prove y_k (or a subseq) converges to a solution of the full problem.