Chapter VI

Theorem of Hahn - Banach

Q1: Given normed space $(X, \|\cdot\|)$, subspace $Y \subset X$

$T \in L(Y, Z)$

$\exists$? extension $\tilde{T} : X \to Z$,

i.e. $\exists$? $\tilde{T}$ s.t. $\tilde{T}|_Y = T$

s.t. $\tilde{T} \in L(X, Z)$?

A: NO for general situations,

- If $\overline{Y} \subset X$: A YES
- Here: If $Z = \mathbb{F}$: A YES.

Q2: If such an extension exists, can we actually get such a $\tilde{T}$ without increasing the norm, i.e.

$\|\tilde{T}\|_{L(X,Z)} = \|T\|_{L(Y,Z)}$?

A:
- $Y \subset X$ dense ✓
- Here $Z = \mathbb{F}$ ✓

Q3: Is such an extension unique?

A: Ch. 4: If $Y \subset X$ dense ✓

Here: X

Recall:

Def. 12: Let $(X, \|\cdot\|_X)$ normed space. Then the dual space of X, is $X^* = L(X, \mathbb{F})$, equipped with operator norm.

Elements of X^* are called bounded linear functionals.

Outline of chapter:

$\mathbb{F} = \mathbb{R}$ $\Longrightarrow$ $\mathbb{F} = \mathbb{C}$

$\boxed{\text{HB on extension of elements of } Y^* \text{ to element of } X^* \text{ (Thm VI.1)}}$ $\Longleftarrow$ $\boxed{\text{HB "general sublinear version" Thm VI.2}}$ $\rightarrow$ Applications

$\boxed{1^{st}\text{ step}}$

2^{nd} step in case of separable spaces using induction

2^{nd} step of proof requires Zorn's Lemma

$\downarrow$

part C

Applications:

"geometric"

$\forall x_0 \in X \setminus \{0\}$ $\exists f \in X^*$ s.t. $f(x_0) \neq 0$

annihilator

Recall:

Any extension F of f will have $\|F\|_{X^*} \geq \|f\|_{Y^*}$.

Conversely we have

$$\|F\|_{X^*} \leq \|f\|_{Y^*}$$

$$\Leftrightarrow \quad |F(x)| \leq p(x) \quad \forall x \in X$$

$$\Leftrightarrow \quad \boxed{F(x) \leq p(x) \quad \forall x \in X}$$

$p(-x) = p(x)$
$F(-x) = -F(x)$

where $p(x) = \|f\|_{Y^*} \cdot \|x\|$

Def. 13: Let X $\mathbb{R}$-vectorspace.

$p: X \to \mathbb{R}$ is called sublinear if

- $p(x+y) \leq p(x) + p(y)$
- $p(\lambda \cdot x) = \lambda p(x)$

$\forall x, y \in X, \quad \lambda \geq 0$

Remark:

- If $\|\text{-}\|_X$ is a semi-norm then $p(x) = M \cdot \|x\|_X$ is sublinear, $M \geq 0$
- If $Y \subset X$ subspace ~~then~~, $(X, \|\text{-}\|)$ n.s. then $p(x) = dist(x, Y)$ is sublinear.

- On $\mathbb{R}^n$

$$p(x) = \operatorname{dist}(x, \mathbb{R}^{n-1} \times (-\infty, 0])$$

$$= \max(x_n, 0).$$

As for $f \in Y^*$ the function

$$p(x) = \|f\|_{Y^*} \cdot \|x\|_X$$

is a seminorm and "all" we require to get Thm 1 is some $F: X \to \mathbb{R}$ s.t.

$$F(x) \leq p(x) \quad \forall x \in X$$

with $F|_Y = f$

we get Thm 1 as a special case of

Thm VI.2 (General "sublinear" version of HB)

Let
- X $\mathbb{R}$-vectorspace
- $p: X \to \mathbb{R}$ sublinear
- $Y \subset X$
- $f: Y \to \mathbb{R}$ linear and

$$f(y) \leq p(y) \quad \forall y \in Y.$$

Then $\exists$ $F: X \to \mathbb{R}$ linear s.t.

$$F|_Y = f \quad \text{and}$$

$$F(x) \leq p(x) \quad \forall x \in X.$$