## Last video:

**Thm VI.2** (Hahn-Banach)

Let
- $X$ $\mathbb{R}$-vectorspace
- $p: X \to \mathbb{R}$ sublinear
- $Y \subset X$ subspace
- $f: Y \to \mathbb{R}$ linear s.t.

  $f(y) \leq p(y) \quad \forall y \in Y$

**Then**: $\exists \; F: X \to \mathbb{R}$ linear

s.t. $F|_Y = f$ and

$F(x) \leq p(x) \quad \forall x \in X.$

## Main step:

**Lemma VI.3** (one-step extension Lemma)

Let:
- $X$ $\mathbb{R}$-vector space
- $p: X \to \mathbb{R}$ sublinear
- $Y, \tilde{Y} \subset X$ subspaces s.t.

  $\exists \, x_0 \in X$ s.t. $\tilde{Y} = \operatorname{span}(Y, x_0)$
- $f: Y \to \mathbb{R}$ linear s.t.

  $f(y) \leq p(y) \quad \forall y \in Y$

**Then**: $\exists \; \tilde{f}: \tilde{Y} \to \mathbb{R}$ linear

s.t. $\tilde{f}|_Y = f$ and

$\tilde{f}(\tilde{y}) \leq p(\tilde{y}) \quad \forall \tilde{y} \in \tilde{Y}.$

so ok provided I can choose

$$v \leq \inf_{u \in Y} p(u+x_0) - f(u).$$

- $\lambda < 0$ : Condition gives

$$v \geq \underbrace{\frac{1}{-\lambda}}_{\frac{1}{|\lambda|}} \left( f(y) - p(y + \lambda x_0) \right)$$

$$\overset{*}{=} f\left(\frac{y}{|\lambda|}\right) - p\left(\frac{y}{|\lambda|} + \underbrace{\frac{\lambda}{|\lambda|}}_{-1} x_0\right)$$

ok if

$$v \geq \sup_{v \in Y} f(v) - p(v - x_0).$$

Get $\exists$ extension provided we show.

<u>Claim:</u> $\inf_{u \in Y} p(u+x_0) - f(u)$

$$\geq$$

$$\sup_{v \in Y} f(v) - p(v - x_0).$$

<u>Proof of Claim:</u>

Let $u, v \in Y$

$$p(u+x_0) - f(u) - (f(v) - p(v-x_0))$$

$$= \underbrace{p(u+x_0) + p(v-x_0)}_{\geq p(u + \cancel{x_0} - \cancel{x_0} + v)} - f(u+v)$$

$$\geq 0 \quad \text{as } u+v \in Y$$

$$f \leq p \text{ on } Y.$$

$\square$ claim $\quad \square$ Lemma.

Proof:

wlog $x_0 \notin Y$ (else $\tilde{Y} = Y$
$\tilde{f} = f$ ✓ )

Hence

$\tilde{Y} = Y \oplus \mathrm{span}(x_0)$

so $\forall \tilde{y} \in \tilde{Y} \quad \exists!\, y \in Y,\ \lambda \in \mathbb{R}$

$\tilde{y} = y + \lambda x_0.$

Note that a $\tilde{f} : \tilde{Y} \to \mathbb{R}$ is linear and s.t. $\tilde{f}|_Y = f$ iff

$$\tilde{f}(y + \lambda x_0) = f(y) + \lambda \cdot \tilde{f}(x_0).$$

$\forall y \in Y \quad \forall \lambda \in \mathbb{R}.$

Q: $\exists$ value $\tilde{f}(x_0) = r$ s.t. this $\tilde{f}$ satisfies $\tilde{f}(\tilde{y}) \leq p(\tilde{y})\ \forall \tilde{y} \in \tilde{Y}$.

Given any $r \in \mathbb{R}$ let

$f_r(y + \lambda x_0) = f(y) + \lambda \cdot r$ so this

gives a linear map

$f_r : \tilde{Y} \to \mathbb{R}.$

To get that $f_r(\tilde{y}) \leq p(\tilde{y})$

need: $f(y) + \lambda \cdot r \leq p(y + \lambda x_0)$

$\forall y \in Y \quad \forall \lambda \in \mathbb{R}$

- If $\lambda = 0$ ✓ as $f(y) \leq p(y)\ \forall y \in Y.$
- If $\lambda > 0$ this is equivalent to

$$r \leq \frac{1}{\lambda} \cdot (p(y + \lambda x_0) - f(y))$$
$$= p\left(\frac{y}{\lambda} + x_0\right) - f\left(\frac{y}{\lambda}\right)$$

$\forall y \in Y \quad \forall \lambda > 0.$

Remark:

Applied in the situation of

- $(X, \|-\|)$ normed space
- $f \in Y^*$, $p(x) = \|f\|_{Y^*} \cdot \|x\|_X$

we hence get that for $Y, \tilde{Y}$ as in lemma 3

$$\exists \tilde{f} \in \tilde{Y}^* \text{ s.t. } \tilde{f}|_Y = f$$

and $\|\tilde{f}\|_{\tilde{Y}^*} = \|f\|_{Y^*}$.

Can use this to give

Proof of Thm VI.1 in case X is separable

Let $Y \subset X$, $(X, \|-\|)$ separable normed space, $f \in Y^*$.

Let $D = \{d_1, d_2, \dots\} \subset X$ dense

Set $Y_0 = Y$

$$Y_{k+1} = \operatorname{span}(Y_k, d_{k+1})$$

$$k = 0, 1, \dots$$

$$f_0 = f$$

Can use 1-step extension to obtain

$$f_{k+1} \in Y_{k+1}^*, \quad f_{k+1}|_{Y_k} = f_k$$

$$\|f_{k+1}\|_{Y_{k+1}^*} = \|f_k\|_{Y_k^*} = \|f\|_{Y^*}$$

Let

$$Y_\infty = \bigcup_{k=1}^{\infty} Y_k$$

and define $f_\infty : Y_\infty \to \mathbb{R}$

by $f_\infty(x) = f_k(x)$ if $x \in Y_k$.

Note: welldefined as for

$$x \in Y_k \cap Y_e = Y_{\min(k,e)}$$

we have $f_k(x) = f_e(x) = f_{\min(k,e)}(x)$.

<u>Note:</u> $f_\infty$ is linear and

$$\|f_\infty(x)\| \leq \|f_k\|_{Y_k^*} \cdot \|x\| = \|f\|_{Y^*} \cdot \|x\|$$

$x \in Y_k \quad \forall k$

so $f_\infty \in Y_\infty^*$ with $\|f_\infty\|_{Y_\infty^*} = \|f\|_{Y^*}$.

As $D$ is dense, also $Y_\infty$ is dense in $X$ so by Thm IV.1

$\exists\, F \in X^*$ s.t.

$$F|_{Y_\infty} = f_\infty \Rightarrow F|_Y = f_\infty|_Y = f$$

and $\|F\|_{X^*} = \|f_\infty\|_{Y_\infty^*} = \|f\|_{Y^*}$

$\boxed{///}$ H.B.