VI.2 Hahn-Banach for $\mathbb{C}$-normed spaces:

Theorem VI.3 (HB for $\mathbb{C}$-case)

Let
- $(X, \|\cdot\|)$ $\mathbb{C}$ normed space
- $Y \subset X$ subspace
- $f \in Y^* = L(Y, \mathbb{C})$

Then $\exists\, F \in X^*$ with $F|_Y = f$

and $\|F\|_{X^*} = \|f\|_{Y^*}$.

Proof:

Given $f \in Y^*$ write

$f = f_1 + i f_2$ where

$f_{1,2} : Y \to \mathbb{R}$.

Note: If f is linear from complex space X into $\mathbb{C}$

then
- $f_{1,2}(x+y) = f_{1,2}(x) + f_{1,2}(y)$ $\forall x, y \in Y$
- For $\lambda \in \mathbb{R}$
  $f_{1,2}(\lambda x) = \lambda \cdot f_{1,2}(x)$

NOT true that $f_{1,2}(ix) = i f_{1,2}(x)$

but

As $f(iy) = i f(y) \quad \forall y \in Y$

$$\begin{array}{ccc} f(iy) & = & i f(y) \\ \| & & \| \\ f_1(iy) + i f_2(iy) & & i \cdot f_1(y) - f_2(y) \end{array}$$

so get additionally that

$$(*) \quad \boxed{f_2(y) = -f_1(iy) \quad \forall y \in Y}$$

Conversely: If f_1, f_2 are $\mathbb{R}$ linear and satisfy $(*)$ then $f = f_1 + i f_2$ is $\mathbb{C}$ linear.

Idea of proof:

① HB $\mathbb{R}$-version
→ extend $f_1 = \mathrm{Re}(f)$ to $F_1 : X \to \mathbb{R}$ $\mathbb{R}$-linear

② "Build" F out of F_1 s.t. F is $\mathbb{C}$-linear

③ Show $F : X \to \mathbb{C}$ is indeed s.t. $\|F\|_{X^*} = \|f\|_{Y^*}$.

As we can view X and Y as $\mathbb{R}$-vector spaces (essentially just dropping the extra operation of multipl. by i)

and as $f_1 : Y \to \mathbb{R}$ is linear

and s.t.

$$|f_1(y)| = |Re(f(y))|$$
$$\leq |f(y)| \leq \|f\|_{Y^*} \cdot \|y\|$$

we get $f_1 \in (Y_{\mathbb{R}})^*$, with
$$\|f_1\|_{(Y_{\mathbb{R}})^*} = \|f\|_{Y^*}$$

Hence by $\mathbb{R}$-version of HB get $\exists F_1 \in (X_{\mathbb{R}})^*$ s.t.

$$F_1|_Y = f_1, \quad \|F_1\|_{(X_{\mathbb{R}})^*} \leq \|f\|_{Y^*}$$

Let now

$$F(x) := F_1(x) - i \cdot F_1(ix).$$

Then F is $\mathbb{C}$-linear

and F is in X^* with $F|_Y = f$

$$\|F\|_{X^*} \leq \|f\|_{Y^*}$$

since

$\forall x \in X$, choose $\theta \in [0, 2\pi]$

s.t. $F(x)e^{i\theta} \in [0, \infty)$

giving

$$|F(x)| = F(x)e^{i\theta} \overset{\mathbb{C}\text{-lin.}}{=} F(e^{i\theta}x)$$
$$= Re(F(e^{i\theta}x)) = F_1(e^{i\theta}x)$$
$$\leq \|F_1\|_{(X_{\mathbb{R}})^*} \cdot \|x\|$$

$\blacksquare$ HB $\mathbb{C}$-case.

VI.3 Applications of HB I:

Let $\mathbb{F} = \mathbb{C}$ or $\mathbb{F} = \mathbb{R}$.

Prop. VI.5:

Let $(X, \|\cdot\|)$ is n.s.

Then $\forall x_0 \in X$, $x \neq 0$

$\exists f \in X^*$ s.t. $\|f\|_{X^*} = 1$

and $f(x_0) = \|x_0\|$.

Proof: Let $Y = \operatorname{span}(x_0)$,

$g : Y \to \mathbb{F}$, $g(\lambda x_0) = \lambda \cdot \|x_0\|$
$\lambda \in \mathbb{F}$

Linear, and

$$\|g\|_{Y^*} = \sup_{\lambda \in \mathbb{F}\setminus\{0\}} \frac{|\lambda \|x_0\||}{\|\lambda \cdot x_0\|} = 1.$$

So as $g \in Y^*$ by HB

$\exists f \in X^*$ s.t.

$\|f\|_{X^*} = \|g\|_{Y^*} = 1$

and $f|_Y = g$ so

$f(x_0) = g(x_0) = \|x_0\|$. $\square$

Cor. VI.7:

Let $(X, \|\cdot\|)$ n.s., $x, y \in X$ s.t.

$x \neq y$.

Then $\exists f \in X^*$ s.t. $f(x) \neq f(y)$.

Proof: Apply Prop 5 to $x - y$.

Cor VI.6:

Let $(X, \|\cdot\|)$ n.s., $X \neq \{0\}$.

we have that $\forall x \in X$

$$\|x\| = \sup_{\substack{f \in X^* \\ \|f\|_{X^*} = 1}} |f(x)|.$$

Proof:

$\forall f \in X^*, \|f\|_{X^*} = 1$

have $|f(x)| \leq \underbrace{\|f\|_{X^*}}_{=1} \cdot \|x\| = \|x\|$

And Prop 5 gives that $\exists f \in X^*$

with $f(x) = \|x\|$, $\|f\| = 1$.

$\blacksquare$

Remark:

$$\|f\|_{X^*} = \sup_{\substack{x \in X \\ \|x\| = 1}} |f(x)|$$

sup not achieved in general

$$\|x\|_X = \sup_{\substack{f \in X^* \\ \|f\| = 1}} |f(x)|$$

sup = max by Prop. 5