VI.4 Geometric applications of Hahn-Banach

Lemma VI.8:

Let X $\mathbb{F}$-vectorspace

Let $f: X \to \mathbb{F}$ s.t. $f \neq 0$.

Then $\forall x_0 \in X$ s.t. $f(x_0) \neq 0$

we have $X = \operatorname{span}\{\ker(f) \cup \{x_0\}\}$

Proof: Given $x \in X$, $\lambda := \frac{f(x)}{f(x_0)}$

Then

$$X = \underbrace{x - \lambda x_0}_{\in \ker(f)} + \lambda x_0$$

since $f(x - \lambda x_0) = f(x) - \frac{f(x)}{f(x_0)} \cdot f(x_0)$

$= 0.$ $\boxed{\checkmark}$

Geometric interpretation ($\mathbb{F} = \mathbb{R}$)

As $\ker(f)$ has codim. $= 1$

we can view

$$\{x \in X : f(x) = \lambda\}, \quad \lambda \in \mathbb{R}$$

as hyperplanes which separate

X into two parts

$$\{x \in X : f(x) < \lambda\}$$

$$\{x \in X : f(x) > \lambda\}.$$

$\underline{Q}$: Given two subsets Ω_1, Ω_2 of which are disjoint

$\exists$? hyperplane that separates Ω_1 and Ω_2

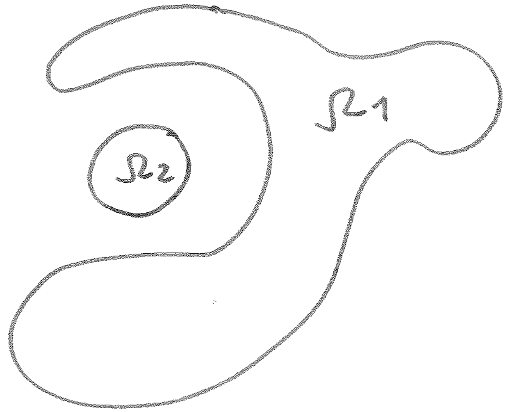
i.e.

$\exists$? $f \in X^*$ s.t. $\exists \lambda \in \mathbb{R}$

$f(x) < \lambda \quad \forall x \in \Omega_1$

$f(x) > \lambda \quad \forall x \in \Omega_2$.

$\underline{A}$: No in general



Suspect: Convexity of sets is important.

Recall $\Omega \subset X$ is convex if $\forall x, y \in \Omega$, $x + t\cdot(y-x) \in \Omega \quad \forall t \in [0,1]$

Remark (→ part c)

If Ω_1, Ω_2 convex subsets of a Banach space and Ω_1 compact, Ω_2 closed

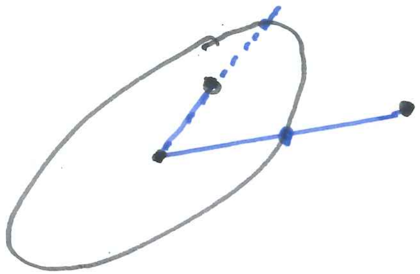
then $\exists f \in X^*$ s.t. $\exists \lambda \in \mathbb{R}$ with

$$f(x_1) < \lambda < f(x_2)$$

$$\forall x_1 \in \Omega_1 \quad \forall x_2 \in \Omega_2.$$

Proof via sublinear vers. of HB.

Given C convex, $0 \in C$



Can show

$$p(x) := \inf \{ \lambda : \tfrac{x}{\lambda} \in C \}$$

is sublinear, $p > 1$ outside of C

≤ 1 inside C.

(off syl.)

Here: Special situation

- separate points $\Omega_1 = \{x\}$, $\Omega_2 = \{y\}$
  $x \neq y$ (→ cor 7)
- separate a point x_0 from a closed subspace Y.
  (→ prop. 9)

Prop VI.9

Let $(X, \|\cdot\|)$ normed space.

$Y \subsetneq X$ subspace, closed.

Then $\forall x_0 \in X \setminus Y$

$\exists f \in X^*$ s.t. $\|f\|_{X^*} = 1$

$f|_Y \equiv 0$ and $f(x_0) = dist(x_0, Y) > 0$.

Proof:

Let $W = span(Y \cup \{x_0\})$

As $x_0 \notin Y$ we have that

$W = Y \oplus span\{x_0\}$

so can write each $w \in W$ as

$w = y + \lambda x_0$ (uniquely)

and can define a linear map

$g : W \to \mathbb{F}$

by $g(y + \lambda x_0) = \lambda \cdot d$

where $d = dist(x_0, Y)$.

Note: $d > 0$ as Y closed

$\Rightarrow \underset{\substack{\Psi \\ x_0}}{X \setminus Y}$ open $\Rightarrow \exists \varepsilon > 0$ $B_\varepsilon(x_0) \subset X \setminus Y$.

Claim: $g \in W^*$ and $\|g\|_{W^*} = 1$.

Note: Once claim is proven

$\rightarrow$ can apply H-B to get $f \in X^*$

$\|f\|_{X^*} = \|g\|_{W^*} = 1$, $f|_Y = 0$, $f(x_0) = d$

as $Y \subset W$, $x_0 \in W$.

Proof of $\|g\|_{W^*} = 1$:

"$\leq$" Given any $w = y + \lambda x_0 \in W$

- if $\lambda = 0$
  $|g(w)| = |0| \leq 1 \cdot \|w\|$ ✓

- if $\lambda \neq 0$ then

$$\|w\| = \| y + \lambda x_0 \|$$
$$= |\lambda| \cdot \| \underbrace{\frac{y}{\lambda}}_{= -\underbrace{\left(-\frac{y}{\lambda}\right)}_{\in Y}} + x_0 \|$$
$$\geq |\lambda| \cdot \operatorname{dist}(x_0, Y)$$
$$= |g(w)|$$

"$\geq$" Let $\varepsilon > 0$. As $d > 0$ we have that

$$d \cdot \frac{1}{1-\varepsilon} > d$$

so $\exists y \in Y$ s.t.

$$\| x_0 - y \| < \frac{d}{1-\varepsilon}$$

Hence $\dfrac{g(x_0 - y)}{\| x_0 - y \|} = \dfrac{d}{\frac{d}{1-\varepsilon}} > 1 - \varepsilon$

so $\|g\|_{W^*} \geq 1 - \varepsilon \quad \forall \varepsilon > 0$

"$\geq$" $\blacksquare$ Claim

$\blacksquare$ Prop. 9.

VI. 5 Further applications of HB (annihilators ...)

Def. 14:

- Given any $A \subset X$, $(X, \|\cdot\|)$ n.s. we define the annihilator of A as $A^\circ = \{f \in X^* : f|_A = 0\}$.

- Given $B \subset X^*$ define
$$B_\circ = \{x \in X : f(x) = 0 \ \forall f \in B\}$$

Remark:

$$B_\circ = \bigcap_{f \in B} \ker(f) = \text{intersect. of closed spaces}$$
so closed subspace

Also easy to check A° is also a closed subspace of X^*.

Prop. VI. 10:

Let $(X, \|\cdot\|)$ n.s., $A \subset X$, $B \subset X^*$.

Then

(i) $\operatorname{span}(A) \subset X$ is dense $\Longleftrightarrow A^\circ = \{0\} \subset X^*$

(ii) $\operatorname{span}(B) \subset X^*$ is dense $\Longrightarrow B_\circ = \{0\} \subset X$.

Proof:

(i) "$\Rightarrow$" If $\operatorname{span}(A) \subset X$ is dense

Then $\forall f \in X^*$ s.t.

$f|_{\operatorname{span}(A)} = 0$

we have $f = 0$

by Lemma IV.2

If $f \in A^\circ$ then

$f(a) = 0 \quad \forall a \in A$

so as f is linear $f|_{\operatorname{span} A} = 0$

so $f = 0$.

Trivially $0 \in A^\circ$ so

$A^\circ = \{0\}$.

"$\Leftarrow$" Suppose $A^\circ = \{0\}$,

$\operatorname{span}(A) \subset X$ not dense.

Then $Y = \overline{\operatorname{span}(A)} \subsetneq X$, closed.

Let $x_0 \in X \setminus Y$ and apply Prop 9

$\to$ gives $f \in X^*$ s.t. $f|_Y = 0$

and $f \neq 0$

But then

- $f \in A^\circ$ since $A \subset Y$
- $f = 0$ $\lightning$

(ii) Suppose $B \subset X^*$ s.t.

$\mathrm{span}(B) \subset X^*$ dense

<u>Claim</u>: $B_0 = \{0\}$

<u>Proof</u>: ✓ if $X = \{0\}$.

Suppose $\exists\, x_0 \in X$, $x_0 \neq 0$ s.t.

$x_0 \in B_0$.

Define $T : X^* \to \mathbb{F}$

$f \mapsto f(x_0)$.

Note: T linear (in f!)

and $|T(f)| = |f(x_0)| \leq \|x_0\| \cdot \|f\|_{X^*}$

so $\|T\|_{L(X^*, \mathbb{F})} \leq \|f\|_{X^*}$

So $T \in (X^*)^*$ and

$T(f) = f(x_0) = 0 \quad \forall f \in B$

as $x_0 \in B_0$

As T is linear also

$T(f) = 0 \quad \forall f \in \mathrm{span}(B)$

as $\mathrm{span}(B) \subset X^*$ is dense, by Lemma IV.2 we get

$f(x_0) = T(f) = 0 \quad \forall f \in X^*$.

<u>But</u>: Prop 5 gives $\exists f \in X^*$ s.t.

$f(x_0) \neq 0$ $\lightning$

▢ (ii)

Lemma VI.11:

Let A any <u>subspace</u> of a normed space $(X, \|\cdot\|)$

Then $\bar{A} = (A^\circ)_\circ$.

Proof:

"$\subseteq$" As $(A^\circ)_\circ$ is closed

and as $A \subset (A^\circ)_\circ$ (check!)

we get $\bar{A} \subset (A^\circ)_\circ$.

"$\supseteq$" Suppose $\exists x \in (A^\circ)_\circ \setminus \bar{A}$

Apply Prop 9 to $\bar{A}$ ← closed subspace $\neq X$ as $(A^\circ)_\circ \setminus \bar{A} \neq \emptyset$.

and to x.

Get $f \in X^*$ s.t.

$f|_{\bar{A}} = 0$ and $f(x) \neq 0$

$\downarrow$ $\qquad\qquad\qquad \downarrow$

$f|_A = 0$ $\qquad\qquad x \notin (A^\circ)_\circ$

i.e. $f \in A^\circ$ $\longrightarrow$

$\lightning$ VI