Chapter VII

Dual spaces, 2^{nd} duals and completion

VII.1 One more property of functionals:

Lemma VII.1

Let $(X, \|\cdot\|)$ n.s., $\underset{\text{linear}}{f : X \to \mathbb{F}}$

Then $\underset{\text{continuous}}{f \in X^*, \text{i.e.}}$ $\iff$ $\underset{\text{closed.}}{\ker(f)}$

⚠ Wrong for

$T : X \to Y$
$\uparrow$
any normed space

in general

Proof:

„$\Rightarrow$" f continuous, $\underset{\square}{\ker(f)} = f^{-1}(\{0\})$

„$\Leftarrow$" wlog $f \neq 0$

So $\exists\, x_0 \in X \setminus \ker(f)$

wlog $f(x_0) = 1$ (else $x_0 \rightsquigarrow \frac{1}{f(x_0)} x_0$)

Let $\delta := \operatorname{dist}(x_0, \ker(f))$

> 0 as $\ker(f)$ closed.

Claim: $\forall x \in X$

$$|f(x)| \leq \delta^{-1} \cdot \|x\| .$$

Proof: ✓ if $x \in \ker(f)$

Let $x \in X$ s.t. $f(x) \neq 0$.

As $f(x_0) = 1$ we have

$$\frac{1}{f(x)} (x - f(x) \cdot x_0) \in \ker(f)$$

So

$$\delta \leq \| x_0 - (\frac{x}{f(x)} - x_0) \|$$

$$= \| \frac{x}{f(x)} \| = \frac{\|x\|}{|f(x)|}$$

$\rightarrow$ $\square$ claim

$\square$ lemma

VII.2 Dual spaces of particular spaces

Goal: Given $(X, \|\cdot\|_X)$

find
- $(Y, \|\cdot\|_Y)$
- $L : Y \to X^*$

s.t. L is bijective and isometric,

i.e. an isometric isomorphism.

Write: $Y \cong (X)^*$

Theorem VII.2 (Riesz-representation Theorem)

Let $(X, (\cdot,\cdot))$ Hilbert space

Then $\forall f \in X^* \quad \exists!\ x_f \in X$

s.t. $f(y) = (y, x_f) \quad \forall y \in Y.$

Furthermore $\|x_f\|_X = \|f\|_{X^*}$

Remark: $\mathbb{F} = \mathbb{R}$

$X^* \ni f \mapsto x_f \in X$

is isometric isomorphism

so $X \cong X^*.$

Proof ($\to$ B4.2 HT)

Rough argument:

As $\ker(f)$ has codim 1 ($\to$ Lemma VI.8)

and is closed ($\to$ Lemma VII.1)

one can prove that

$$X = \ker(f) \oplus \underbrace{(\ker(f))^\perp}_{\substack{\| \\ \{x : (x,y)=0 \ \forall y \in \ker(f)\} \\ = \operatorname{span}(x_0)}}$$

$\to \quad x_f = \lambda \cdot x_0$, suitable $\lambda \in \mathbb{F}$ will work

Thm. VII.3: (Charact. of $(L^p)^*$, $1 \le p < \infty$)

Let $1 \le p < \infty$,

$\Omega \subset \mathbb{R}^n$ any measurable set

Then

$$(L^p(\Omega))^* \cong L^q(\Omega)$$

where $\frac{1}{p} + \frac{1}{q} = 1$

and

$$L : L^q(\Omega) \to (L^p(\Omega))^*$$
$$g \mapsto L_g$$

where $L_g(f) = \int_\Omega f \cdot g \, dx$

$f \in L^p(\Omega)$

is isometric isomorphism.

Proof of Thm 3 \ {surjectivity of L}

← part c

Let $g \in L^q(\Omega)$

Then $L_g : L^p(\Omega) \to \mathbb{F}$

is
- well-defined as, by Hölder, $f \cdot g$ is integrable
- linear (in f) by linearity of $\int$
- $|L_g(f)| = |\int f \cdot g|$

$$\overset{\text{Hölder}}{\le} \|f\|_{L^p} \|g\|_{L^q}$$

So $L_g \in (L^p)^*$ with $\|L_g\|_{(L^p)^*} \le \|g\|_{L^q}$.

So $L : L^q(\Omega) \to (L^p(\Omega))^*$

is
- well defined
- linear (in g) by linearity of $\int$
- $\|Lg\|_{(L^p)^*} \leq \|g\|_{L^q}$

To show:
- L surjective ($\to$ part c)
- $\|Lg\|_{(L^p)^*} \geq \|g\|_{L^q}$ $\forall g \in L^q$. (+)

Proof of (+): Given $g \in L^q(\Omega)$

Let $f = |g|^{q-2} \cdot \bar{g}$

Note: $\frac{1}{p} + \frac{1}{q} = 1 \to p \cdot q = p + q$

$|f|^p = |g|^{(q-1) \cdot p} = |g|^q$ so

$f \in L^p(\Omega)$ with

$\|f\|_{L^p} = \left(\int |f|^p\right)^{1/p} = \left(\int |g|^q\right)^{1/q \cdot q/p}$

$= \|g\|_{L^q}^{q/p}$

Also $Lg(f) = \int |g|^q = \|g\|_{L^q}^q$

so $\frac{Lg(f)}{\|f\|_{L^p}} = \|g\|_{L^q}^{q \cdot (1 - 1/p)} = \|g\|_{L^q}$

$\boxed{//}$

Thm VII.4: (Characterisation of $(\ell^p)^*$, $p<\infty$)

Let $1 \leq p < \infty$,

$$\frac{1}{p} + \frac{1}{q} = 1.$$

Then $(\ell^p)^* \cong \ell^q$

via

$$L: \ell^q \to (\ell^p)^*$$
$$x \mapsto Lx$$

$$Lx(y) = \sum_{j=1}^{\infty} x_j y_j .$$

Proof: Part 1: L well defined and isometric

$\to$ DIY, adjusting previous proof

Here:

Claim 1: L is surjective.

Proof: Let $f \in (\ell^p)^*$

Set $x_j := f(e^{(j)}) \in \mathbb{F}$

(well def. as $e^{(j)} \in \ell^p$)

Claim 2: $x = (x_j) \in \ell^q$ and

$$\|x\|_{\ell^q} \leq \|f\|_{(\ell^p)^*}.$$

Proof of Claim 2:

Enough to show that $\forall N \in \mathbb{N}$

$$\left(\sum_{j=1}^{N} |x_j|^q \right)^{1/q} \leq \|f\|_{(\ell^p)^*}$$

Let $N \in \mathbb{N}$ any fixed number.

Let $$y = \sum_{j=1}^{N} |x_j|^{q-2} \overline{x_j}\, e^{(j)} \in \ell^p$$

Then

- $$f(y) \underset{\uparrow}{=} \sum_{j=1}^{N} |x_j|^{q-2} \overline{x_j} \cdot \underbrace{f(e^{(j)})}_{= x_j}$$

  f linear

  $$= \sum_{j=1}^{N} |x_j|^q$$

- $$\|y\|_{\ell^p} = \left(\sum_{j=1}^{N} \left(|x_j|^{q-1}\right)^p \right)^{1/p}$$

  $$= \left(\sum_{j=1}^{N} |x_j|^q \right)^{1-\frac{1}{q}}$$

$\frac{1}{p} + \frac{1}{q} = 1$

Hence either $\|y\|=0$ -> done

or else

$$\|f\|_{(\ell^p)^*} \geq \frac{f(y)}{\|y\|_{\ell^p}} = \left(\sum_{j=1}^{N} |x_i|^q \right)^{1-(1-\frac{1}{q})}$$

$$= \left(\sum_{j=1}^{N} |x_i|^q \right)^{1/q}$$

$\square$ claim 2

Proof of Claim 1:

To show $f = Lx$, note that claim 2 allows me to talk about Lx as $x \in \ell^q$, and first part of proof tells me $Lx \in (\ell^p)^*$

Note by construction

$$f(y) = Lx(y)$$

$$\forall y \in \{e^{(j)}, j \in \mathbb{N}\}$$

so by linearity of f and of Lx

$$\forall y \in \operatorname{span}\{e^{(j)}, j \in \mathbb{N}\}$$

so by • $\operatorname{span}\{e^{(j)}\} \subset \ell^p$ dense ($p < \infty$!)

• f, Lx <u>bounded</u> lin op.

also get

$$f(y) = Lx(y) \quad \forall y \in \ell^p$$

by Lemma IV.2.

$\boxed{\checkmark}$

<u>Remark:</u>

Proof does not apply for $p = \infty$ case as $\operatorname{span}(\{e^{(j)}\}) \subset \ell^\infty$ NOT dense.

$\rightarrow$ proof of surjectivity does not apply.

Rest is ok, $L : x \to Lx \in (\ell^\infty)^*$
$\quad\quad\quad\quad\quad\quad\; \overset{\cap}{\ell^1}$

is isometric but not surjective.

Example of $f \in (\ell^\infty)^* \setminus L(\ell^1)$

Let $c = \{ x \in \ell^\infty$ s.t. (x_j) converges$\}$

Then $g : c \to \mathbb{F}$

defined by $g(x) = \lim_{j \to \infty} x_j$

well defined, $g \neq 0$, linear

bounded by $|g(x)| \leq \|x\|_{\ell^\infty}$.

So as $g \in (c, \|\cdot\|_{\ell^\infty})^*$

we can apply HB to $f \in (\ell^\infty)^*$ s.t.

$f|_c = g$.

There can't be any $x \in \ell^1$

s.t.

$$f(y) = \sum_{j=1}^{\infty} x_j y_j$$

as e.g. $y = e^{(j)} \in c$,

$$g(e^{(j)}) = 0$$

but ~~so~~ $\sum_{j=1}^{\infty} x_j y_j = x_j = 0$

so $Lx = 0 = f$

$\lightning$ $g \neq 0$

so $f \neq 0$.