VII.3 Second dual & completion

<u>Note</u>: ∃ canonical way / map from a space X to the corresponding dual space in general (except for Hilbertspaces)

- ∃ canonical map

$$X \to X^{**}$$

assinging to each $x \in X$

the map

$$\iota(x) : X^* \to \mathbb{F}$$

that evaluates $f \in X^*$ at the given $x \in X$.

I.e. define canonical map

$$\iota : X \to X^{**}$$

by $\iota(x)(f) = f(x) \quad \forall f \in X^*$

$\forall x \in X$.

<u>Prop. VII.5</u>:
Let $(X, \|\cdot\|)$ n.s. Then the canonical map $\iota : X \to X^{**}$

is

- linear
- isometric.

Def.: A space $(X, \|\ \|)$ is called reflexive if the canonical map $\iota: X \to X^{**}$ is surjective (and hence $X \cong X^{**}$)

Remark:

Since Y^*, Y any n.s., is complete ($Y^* = L(Y, \underset{\substack{\uparrow \\ \text{complete}}}{\mathbb{F}})$)

we have that

X reflexive $\Longrightarrow$ X Banach.

Remark:

- $L^p(\Omega)$, $\ell^p(\mathbb{F})$ are reflexive if $1 < p < \infty$ (and hence q s.t. $\frac{1}{p} + \frac{1}{q} = 1$ also $< \infty$)

- L^∞, ℓ^∞, L^1, ℓ^1 not reflexive.

Proof of Prop. 5:

Let $x \in X$. Then $\iota(x)$ is

- well-defined
- linear (in f)
- $|\iota(x)(f)| = |f(x)| \leq \|f\|_{X^*} \cdot \|x\|_X$

  so $\|\iota(x)\|_{X^{**}} \leq \|x\|_X$.

So: $\iota: X \to X^{**}$ is well defined

and this is also linear (in x)

so it remains to show

<u>Claim:</u> $\|\iota(x)\|_{X^{**}} \geq \|x\|$

$\forall x \in X$.

<u>Proof:</u> wlog $x \neq 0$

By Prop VI.5:

$\exists F \in X^*$ s.t. $\|F\|_{X^*} = 1$

$F(x) = \|x\|_X$

so $\dfrac{|\overset{F(x)}{\overset{\shortparallel}{\iota(x)(F)}}|}{\|F\|_{X^*}} = \|x\|_X$

so $\|\iota(x)\|_{X^{**}} \geq \|x\|_X$. $\boxed{\checkmark}$

<u>Cor. VII.6:</u>

Let $(X, \|\text{-}\|)$ any normed space.

Then X is isometrically isomorphic

to $(\iota(X), \|\text{-}\|_{X^{**}})$ which

we can view as a <u>dense</u> subspace

of the Banach space

$(Y, \|\text{—}\|_{X^{**}})$

where $Y = \overline{\iota(X)} \subset X^{**}$.

<u>Def:</u> $(Y, \|\text{-}\|_Y)$ is called a

completion of a normed space

if $\exists$ isometric linear map

$L: X \to Y$ s.t. $L(X) \subset Y$

dense.

Remark:

The completion of a n.s. $(X, \|\cdot\|_X)$ is determined upto isometric isomorphisms, i.e.

if, Y_1, Y_2 are Banach

- $L_{1,2} : X \to Y_{1,2}$ isometric s.t. $L_i(X) \subset Y_i$ dense

then $\exists\, I : Y_1 \to Y_2$ isom. isomorphism.

Proof:

$$\begin{tikzcd} & X \arrow[dl, "L_1"'] \arrow[dr, "L_2"] & \\ L_1(X) \arrow[d, phantom, "\cap\ \text{dense}"] & & L_2(X) \arrow[d, phantom, "\cap\ \text{dense}"] \\ Y_1 & & Y_2 \end{tikzcd}$$

consider

$$L_2 \circ L_1^{-1} : L_1(X) \to Y_2$$

$$L_1 \circ L_2^{-1} : L_2(X) \to Y_1$$

As these are isometric, in partic. contin.

$Y_{1,2}$ Banach, domains are dense $\subset Y_{1,2}$

Thm IV. 1 $\exists!$ extension $I : Y_1 \to Y_2$

$J : Y_2 \to Y_1$

$I^{-1} = J$