

VII.3 Second dual & completion

Note: $\exists$ canonical way / map
from a space X
to the corresponding
dual space in general
(except for Hilbert spaces)

- $\exists$ canonical map
 $X \rightarrow X^{**}$
assigning to each $x \in X$

The map

$$L(x) : X^* \rightarrow \mathbb{F}$$

that evaluates $f \in X^*$ at
the given $x \in X$.

i.e. define canonical map

$$L : X \rightarrow X^{**}$$

$$\text{by } L(x)(f) = f(x) \quad \forall f \in X^* \\ \forall x \in X.$$

Prop. VII.5:

Let $(X, \|\cdot\|)$ n.s. Then the canonical
map $L : X \rightarrow X^{**}$

is

- linear
- isometric.

Def.: A space $(X, \|\cdot\|)$ is called reflexive if the canonical map $\mathcal{L}: X \rightarrow X^{**}$ is surjective (and hence $X \cong X^{**}$)

Remark:

Since Y^* , Y any n.s., is complete ($Y^* = L(Y, \mathbb{F})$)
 $\uparrow$
 complete

We have that
 X reflexive $\Rightarrow X$ Banach.

Remark:

• $L^p(\Omega)$, $\ell^p(\mathbb{F})$ are reflexive if $1 < p < \infty$

(and hence q s.t. $\frac{1}{p} + \frac{1}{q} = 1$ also $< \infty$)

• L^∞ , ℓ^∞ , L^1 , ℓ^1
not reflexive.

Proof of Prop. 5:

Let $x \in X$ Then $\mathcal{L}(x)$ is

• well-defined

• linear (in f)

• $|\mathcal{L}(x)(f)| = |f(x)| \leq \|f\|_{X^*} \cdot \|x\|_X$
 so $\|\mathcal{L}(x)\|_{X^{**}} \leq \|x\|_X$.

So: $\mathcal{L}: X \rightarrow X^{**}$ is well defined
and this is also linear (in x)
so it remains to show

Claim: $\|\mathcal{L}(x)\|_{X^{**}} \geq \|x\|$
 $\forall x \in X$.

Proof: wlog $x \neq 0$

By Prop VI.5.

$\exists F \in X^*$ s.t. $\|F\|_{X^*} = 1$

$$F(x) = \|x\|_X$$

so
$$\frac{|\mathcal{L}(x)(F)|}{\|F\|_{X^*}} = \|x\|_X$$

so $\|\mathcal{L}(x)\|_{X^{**}} \geq \|x\|_X$. □

Cor. VII.6:

Let $(X, \|\cdot\|)$ any normed space.

Then X is isometrically isomorphic
to $(\mathcal{L}(X), \|\cdot\|_{X^{**}})$ which
we can view as a dense subspace
of the Banach space

$$(Y, \|\cdot\|_{X^{**}})$$

where $Y = \overline{\mathcal{L}(X)} \subset X^{**}$.

Def: $(Y, \|\cdot\|_Y)$ is called a
completion of a normed space
if $\exists$ isometric linear map
 $L: X \rightarrow Y$ s.t. $L(X) \subset Y$
dense.

Remark:

The completion of a n.s. $(X, \|\cdot\|_X)$ is determined upto isometric isomorphisms, i.e.

if Y_1, Y_2 are Banach

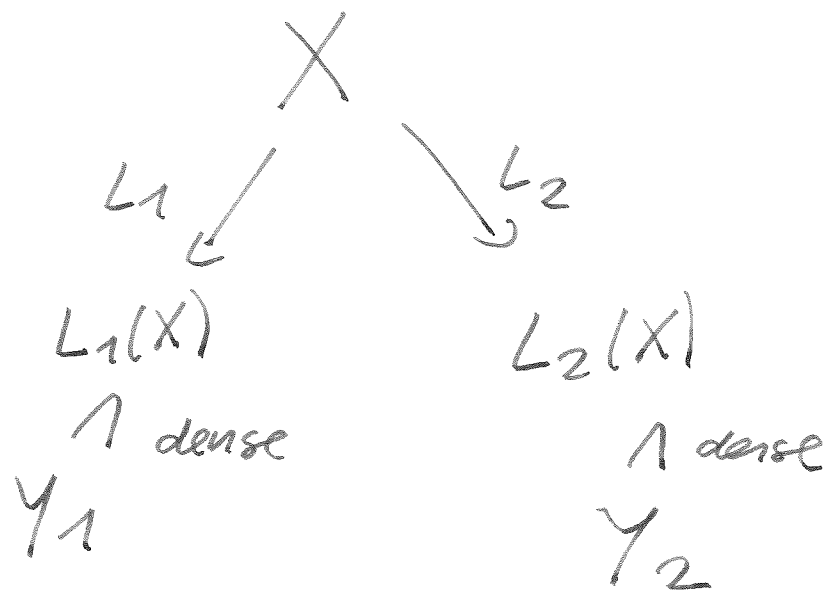
- $L_{1,2}: X \rightarrow Y_{1,2}$ isometric s.t.

$$L_i(X) \subset Y_i \text{ dense}$$

then $\exists I: Y_1 \rightarrow Y_2$

isom. isomorphism.

Proof:



Consider

$$L_2 \circ L_1^{-1}: L_1(X) \rightarrow Y_2$$

$$L_1 \circ L_2^{-1}: L_2(X) \rightarrow Y_1$$

As these are isometric, in partic. contin.

$Y_{1,2}$ @ Banach, domains are dense $\subset Y_{1,2}$

Thm IV.1 $\exists!$ extension $I: Y_1 \rightarrow Y_2$

$$I^{-1} = J$$

$$J: Y_2 \rightarrow Y_1$$