VII. 4 Dual operators

Recall from part A Linear Alg.

Let X, Y vector spaces over $\mathbb{F}$

$X' = \{ L : X \to \mathbb{F} \text{ linear} \}$

$Y' = \{ L : Y \to \mathbb{F} \text{ linear} \}$.

Given any linear $T: X \to Y$ can associate to T the dual operator

$$T' : Y' \to X'$$

defined by

$$\underbrace{(T'f)}_{\in X'} \underbrace{(x)}_{\in X} := \overbrace{f}^{\in Y'} \underbrace{(Tx)}_{\in Y}$$

$f \in Y'$

T' is well def. (in partic. $T'f$ linear (in x))

and T' is linear (in f)

This extends to our setting of bounded lin. operators between normed spaces & their duals.

Prop. $\overline{\text{VII}}$. 7:

Let $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$ are normed spaces.

Let $T \in L(X, Y)$

Then the dual operator

$$T' : Y^* \to X^*$$

$$(T'f)(x) = f(Tx)$$

is well defined, linear and

$$\|T'\|_{L(Y^*, X^*)} = \|T\|_{L(X,Y)}.$$

Using the result from part A

Enough to show:

Claim 1: $T'f \in X^* \quad \forall f \in Y^*$

and $\|T'f\|_{X^*} \leq \|T\|_{L(X,Y)} \|f\|_{Y^*}$

$\rightsquigarrow T' \in L(Y^*, X^*)$, $\|T'\| \leq \|T\|$

and

Claim 2: $\|T'\|_{L(Y^*, X^*)} \geq \|T\|_{L(X,Y)}$

Proof of claim 1:

$\forall x \in X$, $\forall f \in Y^*$

$$|(T'f)(x)| = |f(Tx)|$$
$$\leq \|f\|_{Y^*} \cdot \|Tx\|_Y$$
$$\leq \|f\|_{Y^*} \cdot \|T\|_{L(X,Y)} \cdot \|x\|_X$$

$\boxed{\text{VII}}$ claim 1.

Proof of Claim 2:

Recall $\|T\| = \inf\{M : \|Tx\| \leq M \cdot \|x\| \ \forall x \in X\}$.

Hence enough to show that $\forall x \in X$

$$\|Tx\| \leq \|T'\| \cdot \|x\|.$$

Given $x \in X$

- $Tx = 0 \to$ ✓ i.e. wlog $x \notin \ker(T)$ so $Tx \neq 0$.

By Prop. VI. 5 (Cor. of HB)

$\exists F \in Y^*$ s.t. $\|F\|_{Y^*} = 1$ and $F(Tx) = \|Tx\|$.

So

$$\begin{aligned}\|Tx\| &= F(Tx) \\ &= (T'F)(x) \\ &\leq \|T'F\| \|x\| \\ &\leq \|T'\| \cdot \underbrace{\|F\|}_{=1} \cdot \|x\|\end{aligned}$$

$\boxed{}$ Claim 2

$\boxed{}$ Prop.