Chapter VIII

Spectral theory:

VIII.1 Invertible operators:

Recall: $T \in L(X,Y)$ is invertible if $\exists S \in L(Y,X)$ s.t.

$$TS = Id_Y, \quad ST = Id_X$$

Remark:

T is invertible

$\iff$ T is bijective and algebraic inverse $T^{-1}: Y \to X$ is again a bounded linear operator, i.e. $\exists M$ s.t.

$$\|T^{-1}(y)\|_X \leq M\|y\|_Y \quad \forall y \in Y.$$

$\iff$ T is surjective

and

(*) $\exists \delta > 0$ s.t.

$$\delta \cdot \|x\| \leq \|Tx\| \quad \forall x \in X.$$

Prop. VIII. 1

Let $(X, \|\cdot\|_X)$ be a Banach space

$(Y, \|\cdot\|_Y)$ be a normed space.

Let $T \in L(X, Y)$ s.t. $(*)$ holds.

Then T is injective and $TX \subset Y$ is closed.

Proof: • injective ✓ by. $\ker T = \{0\}$ • T linear

• TX closed:

Let $(y_n) \subset TX$ s.t. $\exists y \in Y$ s.t. $y_n \to y$.

As $y_n \in TX$ $\exists x_n \in X$ s.t. $y_n = Tx_n$ and hence

$$\|x_n - x_m\| \underset{+T \text{ linear}}{\overset{(*)}{\leq}} \delta^{-1} \|\underbrace{Tx_n}_{y_n} - \underbrace{Tx_m}_{y_m}\| \longrightarrow 0$$

as (y_n) converges so is C.S.

So (x_n) is C.S and so as X Banach $\exists x \in X$ s.t. $x_n \to x$ and as T is continuous thus

$$\underset{\substack{\| \\ y_n}}{Tx_n} \to Tx \qquad y_n \to y$$

so $y = Tx \in TX$

$\boxed{\checkmark}$

Note: For X Banach

$T \in L(X,Y)$ invertible

$\iff$ T satisfies $(*)$

and

$TX \subset Y$ is dense.

⚠ For X not complete

2nd part of Prop.

does not apply.

Ex: $Y = (L^1(I), \|\cdot\|_{L^1})$

$I = [0,1]$

$X = (C^0(I), \|\cdot\|_{L^1})$

. . :

$T : C^0(I) \to L^1(I)$

$f \mapsto f$ isometric

$\to$ satisfies $(*)$

but $T(C^0(I)) = C^0(I) \subsetneq L^1(I)$

dense

so not closed.

Remark: (off sylabus)

$T \in L(X,Y) \implies \text{graph}(T) = \{(x, Tx)\} \subset X \times Y$

closed

If X, Y are Banach "$\Longleftarrow$"

closed graph theorem ($\to$ B4.2).

Remark (off syl.)

For X, Y Banach, $T \in L(X,Y)$

T is invertible $\Longleftrightarrow$ T is bijective

(Banachs Isomorphism Thm

or Inverse mapping theorem)

B4.2

Useful result:

Lemma VIII.2:

Let $(X, \|\cdot\|)$ n.s., $S, T \in L(X)$

Assume that ST, TS

are invertible.

Then S and T are invertible.

Proof: Enough to prove

- T surjective

Follows as $TX \supset T(SX) = X$ as TS is invert so surj.

- T satisfies $(*)$.

Suppose not.

Then $\exists\, x_n \in X \setminus \{0\}$ s.t.

$$\frac{\|Tx_n\|}{\|x_n\|} \to 0.$$

Then
$$\frac{\|STx_n\|}{\|x_n\|} \leq \|S\| \cdot \frac{\|Tx_n\|}{\|x_n\|} \to 0$$

so ST violates $(*)$

so ST NOT invertible $\boxed{\text{VII}}$