VIII. 2 Spectrum and resolvent set

For rest of the chapter:

Let $(X, \|\cdot\|)$ be a $\mathbb{C}$-normed space.

Let $T \in L(X) = L(X, X)$

Def. 15:

The spectrum of T

$:= \sigma(T) := \{\lambda \in \mathbb{C} : T - \lambda \mathrm{Id}$ NOT invertible$\}$

$\rho(T)$ = resolvent set

$= \mathbb{C} \setminus \sigma(T)$

$= \{\lambda \in \mathbb{C} : T - \lambda \mathrm{Id}$ is invertible$\}$

For $\lambda \in \rho(T)$, define resolvent operator

$R_\lambda(T) = (T - \lambda \mathrm{Id})^{-1} \in L(X)$.

Note:

$\lambda \in \rho(T) \iff$ we resolve eq

$Tx - \lambda x = b$

uniquely for every $b \in X$ and map

$b \mapsto$ solution

ie. $b \mapsto R_\lambda(T)(b)$

is continuous.

Recall:

$T - \lambda Id$ invertible

$\iff T - \lambda Id$ biject. & $(*)$ holds

$(*)$ $\exists \delta > 0$ s.t.

$$\delta \|x\| \leq \|Tx - \lambda x\|.$$

So $\lambda \in \sigma(T)$ can have different reasons,

- $T - \lambda Id$ can fail to be injective
- $T - \lambda Id$ fails $(*)$
- $T - \lambda Id$ can fail to be surjective while still satisfying $(*)$

Def 16: Let $T \in L(X)$

- $\lambda \in \mathbb{C}$ is an eigenvalue if $\exists\, x \in X \setminus \{0\}$ s.t. $Tx = \lambda x$
- point spectrum of T
  $\sigma_p(T) := \{\lambda \in \mathbb{C} : \lambda \text{ eigenvalue}\}$
- $\lambda \in \mathbb{C}$ is an approximate eigenvalue if $\exists (x_n) \subset X$ s.t. $x_n \neq 0$ and
  $$\frac{\|Tx_n - \lambda x_n\|}{\|x_n\|} \xrightarrow[n \to \infty]{} 0$$
- Approximate point spectrum
  $\sigma_{ap}(T) := \{\lambda \in \mathbb{C} : \lambda \text{ approx eigenvalue}\}$

Remark (off-syl. → B 4.2)

Often one splits $\sigma(T)$ into

- $\sigma_p(T)$
- continuous spectrum
  $\sigma_c(T) = \{\lambda \in \mathbb{C} : T - \lambda Id \text{ injective}, (T-\lambda Id)(X) \subset X \text{ dense}\}$
- residual spectrum
  $\sigma_R(T) = \sigma(T) \setminus (\sigma_p(T) \cup \sigma_c(T))$.

Recall : Lemma II.5 / Cor II.6

Let X Banach, $T_0 \in L(X)$ invertible, $\varepsilon := \frac{1}{\|T_0^{-1}\|} > 0$.

Then $\forall S \in L(X)$ with

$\|S\| < \varepsilon$ we have

$T_0 - S$ is invertible

and

$$(T_0 - S)^{-1} = (Id - T_0^{-1} S)^{-1} T_0^{-1}$$

$$= \sum_{k=0}^{\infty} (T_0^{-1} S)^k T_0^{-1}$$

Applying this for

$T_0 = T - \lambda_0 Id$ for $\lambda_0 \in \varrho(T)$

and for $S = (\lambda - \lambda_0) Id$,

$|\lambda - \lambda_0| < \varepsilon = \frac{1}{\|R_{\lambda_0}(T)\|}$

we get is that

$T - S = T - \lambda Id$ is invertible

so $B_\varepsilon(\lambda_0) \subset \varrho(T)$

and

$$R_\lambda(T) = \sum_{k=0}^{\infty} R_{\lambda_0}(T)^{k+1} (\lambda - \lambda_0)^k$$

Note: For $|\lambda| > \|T\|$ have

$$\|R_\lambda(T)\| = \left\| -\lambda \sum_{j=0}^{\infty} \left(\frac{-T}{\lambda}\right)^j \right\|$$

$$\leq |\lambda| \cdot \sum_{j=0}^{\infty} \left(\frac{\|T\|}{|\lambda|}\right)^j$$

$\|T^j\| \leq \|T\|^j$

$$= \frac{1}{|\lambda| - \|T\|}.$$

Remains to prove

Claim: $\sigma(T) \neq \emptyset$

Proof: By contradiction, use Liouville's thm from $\mathbb{C}$-analysis.

Proof: Assume wrong, in particular $\lambda = 0$ not in $\sigma(T)$.

Thus:
- $T \neq 0$
- T^2 is invertible as T is invertible.

so $T^{-2} \in L(X)$

so by Corollary of Hahn-Banach

$\exists\ f \in (L(X))^*$ s.t. $f(T^{-2}) \neq 0$.

Let $g(\lambda) := f(\underbrace{R_\lambda(T)}_{\in L(X)})$

$\forall \lambda \in \mathbb{C}$

since $\sigma(T) = \emptyset$.

Thm. VIII.3: (Basic properties of spectrum).

Let $(X, \|\cdot\|)$ $\mathbb{C}$-Banach space.

Let $T \in L(X)$.

Then:

(i) $\varrho(T)$ is open and

$\varrho(T) \ni \lambda \mapsto R_\lambda(T) \in L(X)$

is analytic, ie. $\forall \lambda_0 \in \varrho(T)$

$\exists$ nbhd U of λ_0

and $\exists A_j = A_j(\lambda_0, T) \in L(X)$

s.t. $R_\lambda(T) = \sum_{j=0}^{\infty} A_j \cdot (\lambda - \lambda_0)^j$

(ii) $\sigma(T)$ is closed,

we have $\sigma(T) \subset \overline{B_{\|T\|}(0)} \subset \mathbb{C}$

and $\sigma(T) \neq \phi$.

Proof:

(i) done $U = B_\varepsilon(\lambda_0)$, $A_j = R_{\lambda_0}(T)^{j+1}$.

(ii)$_0$ $\sigma(T) = \mathbb{C} \setminus \underbrace{\varrho(T)}_{\text{open by (i)}}$ closed

- For $|\lambda| > \|T\|$

$T - \lambda \mathrm{Id} = \underset{\neq 0}{(-\lambda)} \cdot (\mathrm{Id} - \underbrace{\frac{-T}{\lambda}}_{\|\cdot\| = \frac{\|T\|}{|\lambda|} < 1})$

invert. by lemma II.5.

So $g: \mathbb{C} \to \mathbb{C}$

and $\forall \lambda_0 \in \mathbb{C}$ using the expansion on $U \ni \lambda_0$

$$R_\lambda(T) = \sum_{j=0}^{\infty} A_j (\lambda - \lambda_0)^j$$

and that f is continuous to get that on U

$$g(\lambda) = \sum_{j=0}^{\infty} f(A_j)(\lambda - \lambda_0)^j.$$

So $g: \mathbb{C} \to \mathbb{C}$ is holomorphic.

As expansion around $\lambda_0 = 0$ has $A_1 = T^{-2}$ we get near $\lambda_0 = 0$

$$g(\lambda) = a_0 + \underbrace{a_1}_{\substack{\parallel \\ f(T^{-2}) \neq 0}} \lambda + \sum_{j \geq 2} a_j \lambda^j$$

so g is NOT constant.

BUT: g is bounded on $\mathbb{C}$ and hence by Liouville's thm const. since

- g is bounded on $\overline{B_{2\|T\|}(0)} \subset \mathbb{C}$. compact. as g continuous

- For $|\lambda| \geq 2\|T\| \overset{T \neq 0}{>} \|T\|$

$$\|R_\lambda(T)\| \leq \frac{1}{\|T\|} \text{ so}$$

$$|g(\lambda)| = |f(R_\lambda(T))| \leq \|f\|_{(L(X))^*} \cdot \frac{1}{\|T\|}$$

$\boxed{\text{V}}$