Last time:

For X $\mathbb{C}$-normed space

$T \in L(X)$.

$\sigma(T) := \{\lambda : T - \lambda Id \text{ not invertible}\}$

$\sigma_p(T) = \{\lambda : T - \lambda Id \text{ not injective}\}$

$\sigma_{ap}(T) = \{\lambda : \exists (x_n) \subset X \setminus \{0\}, \frac{\|(T - \lambda Id)(x_n)\|}{\|x_n\|} \to 0\}$

We proved: For X $\mathbb{C}$-Banach space

$\sigma(T) \neq \emptyset$, closed, $\subseteq \overline{B_{\|T\|}(0)}$.

Ex: $(X, \|-\|) = (\ell^1(\mathbb{C}), \|-\|_{\ell^1})$

$$L : X \to X$$
$$(x_1, x_2, \dots) \mapsto (x_2, x_3, \dots)$$

As $\|L\| = 1$ know

$$\sigma(L) \subset \overline{B_1(0)} \subset \mathbb{C}$$

If $\lambda \in \mathbb{C}$ s.t. $|\lambda| < 1$ then

$x = (1, \lambda, \lambda^2, \dots)$ is s.t.

$x \in \ell^1 \setminus \{0\}$ and $Lx = \lambda x$

so $\sigma_p(L) \supset B_1(0)$.

Conversely if $|\lambda| \geq 1$ then above $x \notin \ell^1$ but

$Lx = \lambda x$ would imply that

$x = c \cdot (1, \lambda, \lambda^2, \dots)$

so $x = 0$ $\to$ $\sigma_p(L) \subset B_1(0)$

so $\sigma_p(L) = B_1(0)$.

Note: As
- $\sigma_p(L) \subset \sigma(L)$
- $\sigma(L)$ closed
- $\sigma(L) \subset \overline{B_1(0)}$

get $\sigma(L) = \overline{B_1(0)}$ since

$$\underset{\overline{B_1(0)}}{\overset{\|}{\overline{\sigma_p(L)}}} \subset \overline{\sigma(L)} = \sigma(L) \subset \overline{B_1(0)}$$

so "=" everywhere.

The result that $\sigma(T) \subset \overline{B_{\|T\|}(0)}$ can be refined as follows

[recall $\|T^n\| \leq \|T\|^n$ and for many operators "<"]

Lemma VIII.4

Let
- X - $\mathbb{C}$-Banachspace
- $T \in L(X)$.

If $\lambda \in \sigma(T)$ then

$$\lambda^n \in \sigma(T^n) \quad \forall n \in \mathbb{N}.$$

In particular

$$|\lambda| \leq \|T^n\|^{\frac{1}{n}} \quad \forall \lambda \in \sigma(T)$$

Proof:

Using lemma VIII.2:

ST, TS invert $\Rightarrow$ S, T invert.

In particular if S, T commute and ST invertible $\Rightarrow$ S, T invertible

Note:

$$T^n - \lambda^n Id = (T - \lambda Id)\, S$$

where $S = T^{n-1} + T^{n-2}\lambda + \dots + \lambda^{n-1} Id$

As S and $(T - \lambda Id)$ commute we get:

If $\lambda^n \notin \sigma(T^n)$ then

$(T - \lambda \mathrm{Id})$ invertible i.e. $\lambda \notin \sigma(T)$.

So $\lambda \in \sigma(T) \Rightarrow \lambda^n \in \sigma(T^n)$

$\boxtimes$

Note: Get

$$|\lambda| \leq \inf_{n \in \mathbb{N}} \|T^n\|^{1/n}$$

Indeed $\inf_{n \in \mathbb{N}} \|T^n\|^{1/n} = \lim_{n \to \infty} \|T^n\|^{1/n}$

and :

Thm $\underline{\text{VIII}}$. 5 (without proof)

Let
- X $\mathbb{C}$-Banach space
- $T \in L(X)$.

Then the <u>spectral radius</u>

$$r(T) := \sup \{ |\lambda| : \lambda \in \sigma(T) \}$$

agrees with

$$\lim_{n \to \infty} \|T^n\|^{1/n} = \inf_{n \in \mathbb{N}} \|T^n\|^{1/n}.$$

Thm VIII. 6:

Let
- X $\mathbb{C}$-Banach space
- $T \in L(X)$
- $p(z) = a_0 + a_1 z + \dots + a_n z^n$ any polynomial

Then:

$$p(\sigma(T)) = \sigma(p(T))$$

$p(\sigma(T)) = \{p(\lambda) : \lambda \in \sigma(T)\}$.

$p(T) = a_0 Id + a_1 T + \dots + a_n T^n$

$\ulcorner \quad \sigma(T^n) = \{\lambda^n : \lambda \in \sigma(T)\} \quad \lrcorner$

$\supseteq$ ↑ Lemma 4.

Proof:

Case 1: $p(z) = a_0$

Then $p(T) = a_0 Id$

so $\sigma(p(T)) = \{a_0\}$

As $\sigma(T) \neq \emptyset$, $p(\sigma(T)) = \{a_0\}$ ✓

Case 2: $p(z)$ not constant, ie.

$p(z) = a_0 + \dots + a_n z^n$ for some $n \geq 1$, $a_n \neq 0$.

To show: $\mu \notin \sigma(p(T)) \Leftrightarrow \mu \notin p(\sigma(T))$

Let $\mu \in \mathbb{C}$ and let $\beta_1(\mu), \dots, \beta_n(\mu)$ be the zeros (with multipl.) of

$$p(z) - a_0 = a_n \cdot (z - \beta_1(\mu)) \cdot \ldots \cdot (z - \beta_n(\mu)).$$

Then:

$\mu \notin p(\sigma(T))$

$\Longleftrightarrow \nexists\, \lambda \in \sigma(T)$ s.t.

$\mu = p(\lambda)$ i.e.

s.t. λ is a zero of

$p(z) - \mu$.

$\Longleftrightarrow \forall \lambda \in \sigma(T)\ \nexists\, i$ s.t.

$\lambda = \beta_i(\mu)$

$\Longleftrightarrow \beta_i(\mu) \notin \sigma(T) \quad i = 1, \dots, n$

$\Longleftrightarrow T - \beta_i(\mu) \cdot Id$ is invertible

$\forall i = 1, \dots, n$

$\underset{\text{compos. of invert op. is invertible}}{\Longrightarrow} \overset{\neq 0}{a_n} (T - \beta_1(\mu) Id) \dots (T - \beta_n(\mu) Id)$

is invertible

$\underset{\text{Lemma 2}}{\Longleftarrow}$

i.e. $P(T) - \mu Id$

is invertible

$\Longleftrightarrow \mu \notin \sigma(P(T))$

$\blacksquare$

Thm VIII.7:

Let
- X $\mathbb{C}$-Banach space
- $T \in L(X)$
- $T' \in L(X^*)$ dual operator to T.

Then: $\sigma(T) = \sigma_{ap}(T) \cup \sigma_p(T')$.

Proof: $\sigma_{ap}(T) \subset \sigma(T)$

so enough to show

Claim 1: $\sigma_p(T') \subset \sigma(T)$

Claim 2: $\sigma(T) \setminus \sigma_{ap}(T) \subset \sigma_p(T')$.

Proof of claim 1:

Let $\lambda \in \sigma_p(T')$ so

$\exists f \in X^* \setminus \{0\}$ s.t.

$(T' - \lambda Id)(f) = 0$

i.e. $\forall x \in X$

$$0 = ((T' - \lambda Id)(f))(x)$$
$$= (T'f)(x) - \lambda f(x)$$
$$= f(Tx) - f(\lambda x)$$
$$= f(Tx - \lambda x) \quad \forall x \in X.$$

So $f|_{(T-\lambda Id)(X)} = 0$, $f \neq 0$

we have $T - \lambda Id$ not surjective

so $\lambda \in \sigma(T)$. $\boxtimes$

Recall (Lemma VIII.1)

If $S \in L(X)$ satisfies

(*) $\exists \delta > 0$ s.t. $\|x\| \cdot \delta \leq \|Sx\|$ $\forall x \in X$

then $SX \subset X$ is closed.

<u>Proof of claim 2:</u>

Let $\lambda \in \sigma(T) \setminus \sigma_{ap}(T)$.

<u>Then:</u>

- $T - \lambda Id$ satisfies (*) $\to (T - \lambda Id)(X)$ closed
- $T - \lambda Id$ is not surjective

so $Y = (T - \lambda Id)(X) \subsetneq X$ proper closed subspace.

By Prop. VI.9 (Cor. of HB) we get $\exists f \in X^*$ s.t.

$$f|_Y = 0 \quad , \quad f \neq 0.$$

<u>Note:</u> For every $x \in X$

$$(T'f - \lambda f)(x) = f(Tx) - \lambda f(x) = f(\underbrace{Tx - \lambda x}_{\in Y}) = 0$$

so $T'f = \lambda f$ i.e. $\lambda \in \sigma_p(T')$.

$\boxed{\checkmark}$ claim 2

$\boxed{\checkmark}$