**Professor Joyce** **B3.2 Geometry of Surfaces** **MT 2020**

## Problem Sheet 2

**1.** Let $f : X \to Y$ be a holomorphic map of compact connected Riemann surfaces of degree 1.

**(i)** Show that $f$ has no ramification points.

**(ii)** Show that $f$ is a homeomorphism.

**(iii)** Show that $f^{-1}$ is holomorphic.

**2.** Let $f : X \to Y$ be a nonconstant holomorphic map of compact connected Riemann surfaces, where $X$ is the Riemann sphere. Use the general form of the Riemann-Hurwitz formula to deduce that $Y$ is homeomorphic to $X$.

**3.** The *Korteweg–de Vries equation* which describes shallow water waves is

$$\frac{\partial\phi}{\partial t} + \frac{\partial^3\phi}{\partial x^3} + 6\phi\frac{\partial\phi}{\partial x} = 0.$$

**(i)** A solution with a fixed wave form is given by $\phi(x,t) = f(x - ct)$. Show that $f$ satisfies the equation

$$-cf' + f''' + 6ff' = 0.$$

**(ii)** Using the relation $(\wp')^2 = 4(\wp - e_1)(\wp - e_2)(\wp - e_3)$ find constants $a, b$ such that $f = a\wp + b$ satisfies this equation where $\wp$ is the Weierstrass $\wp$-function. Can you describe the sort of wave this corresponds to?

**4.** Let $f : X \to Y$ be a holomorphic map of compact connected Riemann surfaces of degree 2. Show that there is a non-trivial holomorphic homeomorphism $\sigma : X \to X$ such that $f \circ \sigma = f$ and $\sigma^2$ is the identity map. How many fixed points does your map have?

**5. (The classification of elliptic curves.)** There are bijections

$$\frac{\left\{\begin{array}{c}\text{Riemann surfaces}\\ \text{homeomorphic to a torus}\end{array}\right\}}{\text{biholomorphisms}} \longleftrightarrow \frac{\left\{\begin{array}{c}\text{Quotients}\\ \mathbb{C}/\Lambda \text{ for } \Lambda \text{ a lattice in } \mathbb{C}\end{array}\right\}}{\text{biholomorphisms}}$$

$$\longleftrightarrow \frac{\left\{\begin{array}{c}\text{Quotients}\\ \mathbb{C}/(\mathbb{Z}+\mathbb{Z}\tau) \text{ with } \tau \in \mathbb{H}\end{array}\right\}}{\text{biholomorphisms}} \longleftrightarrow \mathbb{H}/\operatorname{PSL}(2,\mathbb{Z}).$$

Here $\mathbb{H} = \{\tau \in \mathbb{C} : \operatorname{Im}\tau > 0\}$ is the upper half-plane in $\mathbb{C}$. The second map comes by writing $\Lambda = \langle \omega_1, \omega_2 \rangle_{\mathbb{Z}}$, choosing $\tau$ to be whichever of $\omega_2/\omega_1$ or $-\omega_2/\omega_1$ lies in $\mathbb{H}$ and noting that $\Lambda = \omega_1 \cdot (\mathbb{Z} + \mathbb{Z}\tau)$ so $\mathbb{C}/\Lambda \cong \mathbb{C}/(\mathbb{Z}+\mathbb{Z}\tau)$. The second map is $\mathbb{C}/(\mathbb{Z}+\mathbb{Z}\tau) \leftrightarrow [\tau]$, and $\operatorname{PSL}(2,\mathbb{Z}) = \operatorname{SL}(2,\mathbb{Z})/\{\pm I\}$ acts on the upper half-plane $\mathbb{H} = \{z \in \mathbb{C} : \operatorname{Im}(z) > 0\}$ by Möbius transformations, that is, $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ in $\operatorname{SL}(2,\mathbb{Z})$ acts by $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) : z \mapsto (az+b)(cz+d)$.

Although we will not need the following fact, some easy group theory shows that $\operatorname{SL}(2,\mathbb{Z})$ is generated by $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$. The corresponding Möbius maps $S(z) = -1/z$ and $T(z) = z+1$ are rather useful in this exercise.

**(a)** For $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2,\mathbb{Z})$, show that $\operatorname{Im}(Az) = \frac{1}{|cz+d|^2} \cdot \operatorname{Im}(z)$. Deduce that, given a constant $K$, only finitely many $c, d \in \mathbb{Z}$ satisfy $\operatorname{Im}(Az) > K$.

**(b)** Show that $\mathbb{H}/\operatorname{PSL}(2,\mathbb{Z})$ is a topological space homeomorphic to $\mathbb{C}$, by first showing each point of $\mathbb{H}/\operatorname{PSL}(2,\mathbb{Z})$ has a representative inside the "strip"

$$\{\tau \in \mathbb{H} : |\operatorname{Re}(\tau)| \leq 1/2, |\tau| \geq 1\}$$

and then checking that the only remaining identifications are on the boundary of the strip.[1]

**(c)** Show that $\operatorname{PSL}(2,\mathbb{Z})$ acts freely[2] on $\mathbb{H}$ except at the points in the $\operatorname{PSL}(2,\mathbb{Z})$-orbits of $e^{\pi i/3}$ and of $i$, and show that the stabilisers of those points are respectively $\mathbb{Z}/3$ and $\mathbb{Z}/2$.

**(d)** Briefly comment on why the natural local complex coordinate from $\mathbb{H}$ makes $\mathbb{H}/\operatorname{PSL}(2,\mathbb{Z})$ into a Riemann surface except at $e^{\pi i/3}$ and $i$.

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[1] Hint. Try to maximize the imaginary part for the orbit of $z$ under the action.

[2] A group $G$ acts freely on $X$ if stabilizers are trivial, explicitly: if $g \bullet x = x$ for some $x$, then $g = 1$.