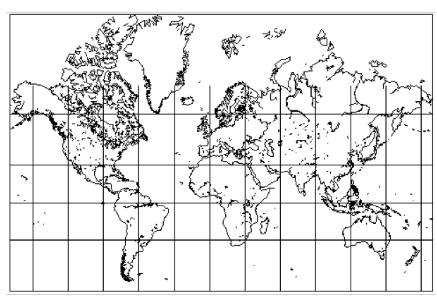
Professor Joyce **B3.2 Geometry of Surfaces** **MT 2020**

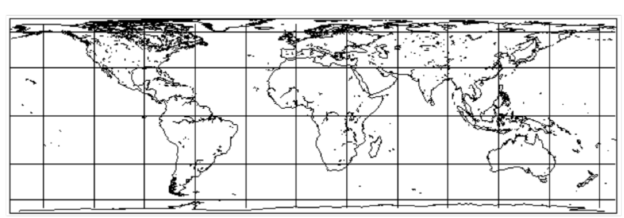
Problem Sheet 3

1. Let U be an open subset of $\mathbb{R}^2$ and let $\mathbf{r} : U \to \mathbb{R}^3$ be a smooth parametrisation of a surface $S = \mathbf{r}(U) \subseteq \mathbb{R}^3$. Let $E\mathrm{d}u^2 + 2F\mathrm{d}u\mathrm{d}v + G\mathrm{d}v^2$ be its first fundamental form. A parametrisation is said to be *conformal* if it preserves angles between intersecting curves, and *equiareal* if it preserves areas.

- Show that the parametrisation is *conformal* if and only if $E = G$ and $F = 0$, and is *equiareal* if and only if $EG - F^2 = 1$.
- What is the first fundamental form of the spherical coordinates local parametrisation of the unit sphere, given by $\mathbf{r}(\theta, \phi) = (\cos\theta\cos\phi, \sin\theta\cos\phi, \sin\phi)$? Show that this parametrisation is neither conformal nor equiareal. (In this familiar parametrisation, θ gives the longitude and ϕ the latitude.)
- *Mercator's projection* of the unit sphere minus the Date Line takes a point $\mathbf{r}(\theta, \phi)$ with latitude ϕ and longitude θ to $(\theta, \log\tan(\frac{\phi}{2} + \frac{\pi}{4}))$ in $(-\pi, \pi) \times \mathbb{R}$. Show that this parametrisation is conformal but not equiareal.



- *Lambert's cylindrical projection* takes a point $\mathbf{r}(\theta, \phi)$ with latitude ϕ and longitude θ to $(\theta, \sin\phi)$.



Show that this parametrisation is equiareal.

2. The *tractrix* is a curve in $\mathbb{R}^2$ such that the distance along any tangent line from its point of contact with the curve to its point of intersection with the x-axis is 1. If θ is the angle the tangent line makes with the x-axis, show that the surface of revolution (the tractoid) obtained by rotating the tractrix about the x-axis has first fundamental form $\cot^2\theta\,\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}v^2$, where v is the angle of rotation of the surface of revolution. By making a suitable change of coordinates between (v, θ) and (x, y), show that the tractoid is locally isometric to the *hyperbolic plane* with first fundamental form $(\mathrm{d}x^2 + \mathrm{d}y^2)/y^2$.

3. Show that the Gaussian curvature of a surface which is the graph of a smooth function $z = f(x, y)$ is given by

$$K = \frac{f_{xx}f_{yy} - f_{xy}^2}{(1 + f_x^2 + f_y^2)^2}.$$

Calculate K when $f(x, y) = xy$ and sketch the surface.

4. Let $\mathbf{r}(u, v)$ be a parametrized surface in $\mathbb{R}^3$ with $(u, v) \in U$, a connected open set in $\mathbb{R}^2$. Let S^2 denote the sphere of radius 1 with centre the origin in $\mathbb{R}^3$ and let $\mathbf{n} : U \to S^2$ be the mapping defined by assigning to each point of the surface the unit normal. Suppose that the restriction of $\mathbf{n}$ to U is a bijection onto $\mathbf{n}(U)$ and that the Gaussian curvature K is nowhere zero in U. Show that the area of $\mathbf{n}(U)$ equals the absolute value of $\int_U K dA$.

5. Let S be the unit sphere in $\mathbb{R}^3$ and γ the circle obtained by intersecting S with the plane $z = \sqrt{1 - a^2}$. Calculate the geodesic curvature of γ and the area of the smaller region of the sphere bounded by γ, and use these results to illustrate the Gauss–Bonnet theorem.