Professor Joyce **B3.2 Geometry of Surfaces** **MT 2020**

Problem Sheet 4

1. The smooth function $f : \mathbb{R}^2 \to \mathbb{R}$ is given by $f(x, y) = \cos 2\pi x + \cos 2\pi y$. Determine and classify the critical points of f.

A torus T is formed by identifying opposite edges of $[0, 1] \times [0, 1]$ so that f induces a smooth function on T. Use it to verify that $\chi(T) = 0$.

2. Prove that along a geodesic γ on a surface of revolution the product $\rho \sin \varphi$ is constant, where $\rho(s)$ is the distance from $\gamma(s)$ to the axis of revolution, and $\varphi(s)$ is the angle between $\gamma'(s)$ and the meridian through $\gamma(s)$.

Prove that on the ellipsoid of revolution obtained by rotating $x^2/a^2 + y^2/b^2 = 1$ about the x-axis, every geodesic which is not a meridian remains always between two parallels of latitude.

[On a surface of revolution $\boldsymbol{r}(u, v) = (u, f(u) \cos v, f(u) \sin v)$ the meridians are given by v =constant and the parallels of latitude by u =constant].

3. Let $\mathcal{H}$ be the upper half plane model of the hyperbolic plane and let L be a geodesic in $\mathcal{H}$. Find the locus of all points equidistant from L.

[Hint: First consider the geodesic $\{(0, e^{-t}) : t \in \mathbb{R}\}$ and find the images of a point P with respect to all isometries mapping the geodesic to itself.]

4. A hyperbolic triangle has angles α, β, γ, respectively, and opposite sides of lengths a, b, c, respectively. By using the *hyperbolic "cos" formula*

$$\cosh c = \cosh a \cosh b - \sinh a \sinh b \cos \gamma,$$

applied to relevant right angled triangles, or otherwise, show that

$$\frac{\sinh a}{\sin \alpha} = \frac{\sinh b}{\sin \beta} = \frac{\sinh c}{\sin \gamma}.$$

5. Show that if a hyperbolic triangle is right-angled, with $\gamma = \pi/2$, then $\cosh c = \cosh a \cosh b$ and use this to prove that in a hyperbolic triangle the length c of the hypotenuse is always longer than the corresponding Euclidean result $\sqrt{a^2 + b^2}$.